INTERNALITY IN FAMILIES

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Abstract

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In geometric stability theory, internality plays a central role, as a tool to explore the fine structure of definable sets. Studying internal types, one often encounters uniformly defined families of internal types. This thesis is mainly concerned with the study of such families, both from an abstract model-theoretic perspective and in concrete examples.

In the first part, we generalize one of the fundamental tools of geometric stability theory, type-definable binding groups, to certain families of internal types, which we call relatively internal. We obtain, instead of a group, a type-definable groupoid, as well as simplicial data, encoding structural properties of the family.

In the second part, we introduce a new strengthening of internality, called uniform internality. We expose its connection with the previously constructed groupoids, and prove it is a strengthening of preserving internality, a notion previously introduced by Moosa. We then explore examples in differentially closed fields and compact complex manifolds.

In the last part, we study a structural feature of stable theories, called the canonical base property. We prove that Hrushovski, Palacín and Pillay's counterexample does not transfer to positive characteristic. Elaborating on the counterexample, we also provide an abstract configuration violating the canonical base property.

You have to dig it to dig it, you dig?

Thelonious Monk

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CHAPTER 1

INTRODUCTION

1.1 Background

One of the foundational themes of modern model theory is to study complete first-order theories via the fine structure of their definable sets. This approach started in the sixties with Morley's work [26], where he proves that a complete countable theory categorical in one uncountable cardinal is categorical in all uncountable cardinal. To do so, he introduces what will become known as Morley rank, a notion of dimension for definable sets.

In the same vein, Zilber proved in 1980 ([41]) that totally categorical theories are not finitely axiomatizable. To do so, he introduced internal definable sets. Given two definable sets X and Y, the set X is *internal* to Y if (roughly) there is a definable bijection between X and a definable subset of Y^{eq}. Of crucial importance is the possibility of needing new parameters to define this bijection.

In this same paper, Zilber gave the first construction of a binding group. If X is internal to Y, the binding group is a definable group acting on X, which is isomorphic, as a group action, to the group of automorphisms of X fixing Y pointwise. These can be seen as generalizing Galois groups for field extensions. Supporting this analogy is Poizat's observation that Kolchin's differential Galois groups are an incarnation of binding groups [33].

Internal sets and their binding groups proved to be extremely powerful tools in the understanding of stable theories of finite Morley rank. The working philosophy is that models can in some sense be recovered from their strongly minimal sets, using internal sets. More precisely, Zilber's Ladder Theorem [42] states that any definable set is obtained as an iterated fibration, where each of the fiber is internal to a strongly minimal set.

A few years later, internal types make their first appearance. They are used by Hrushovski in [10] to prove that unidimensional theories are superstable. Crucial to this proof is the construction of the binding group of a type, which is type-definable. Moreover, in Hrushovski's thesis [9], Zilber's Ladder Theorem is transfered to types, in what is known as the analysis of a type: if $a \models p$ and $\mathcal{P}$ is a fixed family of partial types, then a $\mathcal{P}$-analysis of p is a sequence $a = a_n, \cdots, a_0$ such that $\mathrm{tp}(a_{i+1}/a_i)$ is internal to $\mathcal{P}$ for all i. In the spirit of the Ladder Theorem, one can prove that in a superstable theory of finite U-rank, any type is analysable in the family of U-rank one types. Let us also mention Buechler's notion of levels, introduced in [1]. It is very closely related to analysability, and used, in the same paper, to prove Vaught's conjecture for superstable theories of finite rank.

Hence, analysability is an omnipresent phenomenon in superstable theories, but it is still poorly understood. In analogy with internal types and binding groups, it is natural to ask what is the definable algebraic structure, if any, arising from an analysable type. This question was asked, and partially answered, for analysable covers in Hrushovski's [13] and Haykazyan and Moosa's [8]. The heart of this dissertation is the construction of various type-definable groupoids arising from analysable types.

In the late 2000's, Moosa and Pillay [25], motivated by complex analytic geometry, introduced a strong structural feature of superstable theories of finite rank: the canonical base property. It has a number of attractive consequences, such as a simpler proof of Zilber's trichotomy for differentially closed fields of characteristic zero given by Pillay and Ziegler in [32]. For a few years, it was conjectured that all superstable theories of finite rank had the canonical base property, until a counterexample was constructed by Hrushovski, Palacín and Pillay in [16]. The last chapter of this dissertation will be concerned with elaborating on their construction.

Roughly speaking, the canonical base property states that if b is the canonical base of $\mathrm{tp}(a/b)$, then $\mathrm{tp}(b/a)$ is internal to the family of non-locally modular types of U-rank one. By the work of Chatzidakis [2], this type is always analysable in the family of non-locally modular rank one types. Thus, uncovering criteria for an analysable types to be internal is a very natural approach to studying the canonical base property.

This motivates Moosa's introduction [24] of preservation of internality, a property of an internal type that, in particular, forces certain analysable types to be internal. Our work on groupoids will lead us to introduce a stronger property, which we call *uniform relative internality*, that exactly captures how analysable types can be forced to be internal.

1.2 Summary of Results

In Chapter 3, we set up the main theme of this dissertation: relatively internal types. These are, grosso modo, types equipped with a definable map, such that all fibers are internal. More precisely, let us fix a family of partial types $\mathcal{P}$, all over $\emptyset$, we define:

Definition. Let q be a complete type over A, and π an A-definable partial map, defined on any realization of q. The pair (q, π) is said to be relatively $\mathcal{P}$-internal if for any (some) $a \models q$, the type $\mathrm{tp}(a/\pi(a)A)$ is stationary and $\mathcal{P}$-internal.

The main structural result of Chapter 3 is the generalization to relatively internal types of the binding group construction:

Theorem 3.1.3. Let (q, π) be a relatively internal pair. There is an A-type definable groupoid $\mathcal{G}$, acting A-definably on $q(\mathbb{M})$ as partial elementary maps between fibers of π restricted to $q(\mathbb{M})$, fixing $\mathcal{P}(\mathbb{M}) \cup A$, where $\mathcal{P}(\mathbb{M})$ is the set of realizations of $\mathcal{P}$ in the monster model.

We refer the reader to Chapter 3 for more details on this groupoid. The rest of the Chapter is dedicated to connecting properties of (q, π) to properties of the groupoid. The first groupoid property studied is retractability, which was introduced in [7]. It states that there is an A-definable full and faithful functor from $\mathcal{G}$ to an A-definable group. This is very strong, and as such implies a strong restriction on q: it has to be the product of two types, one of which is $\mathcal{P}$-internal and π is a projection from q onto the other. An application to pullbacks in differential fields of characteristic zero is discussed.

We then move to a more complex object, encoding how fibers interact with each other: a Delta groupoid (see Definition 3.3.15 and the following proposition). Using Delta groupoids, we state an equivalence between internality of q and a certain projective limit of type-definable groups being itself type-definable. Roughly speaking, we can construct a projective system of type-definable groups by considering the binding groups of the relatively $\mathcal{P}$-internal pairs $(q^{\otimes n}, \pi)$, i.e. the binding groups of n independent fibers, for all $n \in \mathbb{N}$. If $\pi(q)$ is $\mathcal{P}$-internal, then the type q is $\mathcal{P}$-internal if and only if the restrition maps between these groups become injective if n is large enough (implying type-definablity of the projective limit).

Moving to Chapter 4, we stop working with groupoids and instead give a condition equivalent to the internality criteria of Chapter 3, namely:

Definition. Let (q, π) be a relatively $\mathcal{P}$-internal pair, with $q \in S(\emptyset)$. Then (q, π) is said to be uniformly relatively $\mathcal{P}$-internal (resp. almost $\mathcal{P}$-internal) if there is a tuple e such that for any $a \models q$, we have $a \in \mathrm{dcl}(\pi(a), e, \mathcal{P})$ (resp. $a \in \mathrm{acl}(\pi(a), e, \mathcal{P})$).

We start by giving various equivalent characterization of this property, inspired by similar statements for internal types. We also give an interesting structural result:

Theorem 4.1.9. Suppose (q, π) is uniformly relatively almost internal. Then there is a tuple of parameters t such that:

1. for any $a \models q$ independent from t over $\emptyset$, we have $a \in \mathrm{acl}(t, \pi(a), \mathcal{P})$
2. $\mathrm{tp}(t/\emptyset)$ is $\mathcal{P}$-internal

We then prove that uniform internality implies preservation of internality:

Definition. The stationary type $\mathrm{tp}(a/b)$ is said to preserve $\mathcal{P}$-internality if for any c such that $\mathrm{tp}(b/c)$ is internal, the type $\mathrm{tp}(a/c)$ is internal.

We then set out to determine if and when this implication is strict. To start with, we exhibit examples of types preserving internality to the constants but not uniformly internal to them, in differentially closed fields of characteristic zero. We then consider two examples of preservation of internality from the literature:

- Differential tangent bundles in DCF_0
- Moishezon morphisms in compact complex manifolds

In the case of compact complex manifolds, we show that uniform internality is much stronger than preserving internality, as it implies algebraicity over the projective line. In the case of differential tangent bundles, we only present a few uniformly internal examples. In a forthcoming paper, joint with Rémi Jaoui and Anand Pillay, a non-uniformly internal example is constructed.

Finally, Chapter 5 is dedicated to the canonical base property (CBP):

Definition. Let T be a superstable theory and $\mathbb{M} \models T$ a monster model. The theory T is said to have the canonical base property if (possibly working over some parameters) for any tuples $a, b \in \mathbb{M}$, if $\mathrm{stp}(a)$ has finite Lascar rank and $b = \mathrm{Cb}(\mathrm{stp}(a/b))$, then $\mathrm{tp}(b/a)$ is almost $\mathcal{P}$-internal, where $\mathcal{P}$ is the family of non-locally modular rank one types.

We prove that Hrushovski, Palacín and Pillay's counterexample [16] does not go through in positive characteristic, answering a question of these same authors. Abstracting their counterexample, we provided an axiomatic framework forcing a theory to not have the canonical base property, with the hope that it could be used to construct new counterexamples.

CHAPTER 2

PRELIMINARIES

2.1 Stability Theory

Throughout this dissertation, we will assume familiarity with standard model theoretic notions such as definable sets, types, automorphisms, algebraic and definable closures... A reference for these is [38]. Our notation will be standard. We will use this preliminary section to discuss more specific stability machinery that we will use regularly. All results and definitions from this section are taken from [38] and [29].

But first, let us introduce a convenient set up: the *monster model*. In their daily life, the working model-theorist quickly realize how frequently they have to choose a model, which has to be saturated enough, homogeneous enough, and contain all parameters in use. This sort of argument tends to become tedious, and one truly wishes to fix such a model once and for all. Doing so requires, in general, to assume more than just the standard ZFC axioms, and said model would have to be class-size.

Let us be more concrete: let T be any complete first order theory with infinite models. Following [38], Chapter 6, Section 1, we will work in the extension BCG (Bernays-Gödel+Global Choice) of ZFC, which adds classes to ZFC (we refer to [38], Appendix A for more precision). Under this set up, there is a unique (up to isomorphism) class size model $\mathbb{M} \models T$ such that:

- any type over any subset of $\mathbb{M}$ is realized in $\mathbb{M}$
- $\mathbb{M}$ is κ-saturated for any cardinal κ
- any model of T elementary embeds in $\mathbb{M}$

- any elementary bijection between two subsets of $\mathbb{M}$ can be extended to an automorphism of $\mathbb{M}$

We call $\mathbb{M}$ the monster model of T. In this dissertation, we will usually work in the monster model of a stable theory. In particular, all sets of parameters, realizations of tuples... will be taken in $\mathbb{M}$, and all models will be elementary substructures of $\mathbb{M}$.

Moreover, all (type-)definable sets will be seen as contained in $\mathbb{M}$. Note that if infinite, then they actually are class-sized. In this dissertation, we will call any set-sized set *small* (the usual tools of model theory can be used freely over that set), and a class-size set *big* (more care will have to be taken). The need to work over big sets will arise frequently. We will discuss how to do so rigorously in the next section.

Finally, let us recall that the use of monster models, although extremely convenient, is optional, and that all our proofs would go through by choosing appropriately large models. We refer to [38] for details on how to avoid them.

From now on, when fixing a theory T, we will also always, in the background, fix a monster model $\mathbb{M}$, where all tuples, parameters,... will live.

Let us now start on stability theory. Fix some complete theory T in a language $\mathcal{L}$. An $\mathcal{L}$ formula is said to be *stable* if there is an infinite cardinal λ such that $|S_\phi(A)| \leq \lambda$ whenever $|A| \leq \lambda$. Here, the set $S_\phi(A)$ is the space of ϕ-types over A, i.e. maximal consistent sets of formulas of the form $\phi(x, a)$ or $\neg\phi(x, a)$, for some $a \in A$. The theory T is said to be stable if all its formulas are.

Stability was introduced by Shelah [36], with the goal studying the number of non-isomorphic models of a theory T. Many more combinatorial and/or geometric characterizations have been found since. We will only cite the ones that will be useful to us. First, there is *definability* of types:

Definition 2.1.1. A type $p(x) \in S(A)$ is said to be definable over $B \subset A$ if for any $\mathcal{L}$-formula $\phi(x, y)$ there is an $\mathcal{L}(B)$ formula $\psi(y)$ such that for any $a \in A$, we have:

$$\phi(x, a) \in p \text{ if an only if } \models \psi(a)$$

It is said to be definable if it is definable over A.

The formula $\psi(y)$ is called a *defining scheme* for $\phi(x,y)$. A common use of definable types is to identify some set with the defining scheme of a definable type, in order to prove that said set is definable. We will use that strategy in the proof of Theorem 3.1.3.

We have the following characterization of stability:

Theorem 2.1.2. *The theory T is stable if and only if all types are definable.*

For the reminder of this section (and dissertation), we will always restrict ourselves to stable theories. Thus from now on, we work in the monster model $\mathbb{M}$ of a stable theory T.

A key feature of stable theories is the existence of a well-behaved notion of independence, called non-forking:

Definition 2.1.3. A family of formulas $(\psi_i(x))_{i\in I}$ is k-inconsistent, for $k \in \mathbb{N}$, if for any k-element subset K of I, the set $\{\psi_i(x), i \in K\}$ is inconsistent.

A formula $\phi(x,a)$ divides over A if there is a sequence of realizations $(a_i)_{i<\omega}$ of $\mathrm{tp}(a/A)$ and $k \in \mathbb{N}$ such that $\{\phi(x,a_i), i < \omega\}$ is k-inconsistent. A partial type $\Phi(x)$ divides over A if it contains a formula that does.

Definition 2.1.4. A partial type $\Phi(x)$ forks over A if there are some formulas $\phi_0(x), \cdots, \phi_n(x)$ such that $\Phi(x)$ implies $\bigvee_{i=1}^{n} \phi_n(x)$ and each ϕ_i divides over A.

If $A \subset B$, and we have $p \in S(A)$ and $q \in S(B)$ extending q, we say that q is a non-forking extension of p if q does not fork over A. Such an extension always exists, for any $B \supset A$.

The reader worried about the extra complexity introduced by forking will be delighted to learn:

Fact 2.1.5. *In stable theories, forking and dividing coincide.*

We are often interested in elements of $\mathbb{M}$ being independent from each other. Non-forking provides us with such a notion:

Definition 2.1.6. Let A be a set of parameters, and $b, c \in \mathbb{M}$. We say that b is independent from c over A if $\mathrm{tp}(b/Ac)$ does not fork over A. This is commonly written as $b \mathop{\smile\!\!\!\!\!\!|}_A c$.

If C is a set of parameters, we similarly write $b \mathop{\smile\!\!\!\!\!\!|}_A C$ if $\mathrm{tp}(b/AC)$ does not fork over A.

Central tools for the study of stable theories are indiscernible sequences and Morley sequences:

Definition 2.1.7. Let A be a set of parameters. A sequence $(a_i)_{i<\omega}$ of tuples is said to be A-indiscernible if for all $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$, we have $\mathrm{tp}(a_{i_1}, \cdots a_{i_n}/A) = \mathrm{tp}(a_{j_1}, \cdots, a_{j_n}/A)$.

A sequence $(a_i)_{i<\omega}$ of tuples is said to be A-independent if $a_i \mathop{\smile\!\!\!\!\!\!|}_A \{a_j, j < i\}$ for all $i < \omega$.

Finally, a sequence $(a_i)_{i<\omega}$ is said to be a Morley sequence over A if it is both A-indiscernible and A-independent.

Fixing a type $p \in S(A)$, a Morley sequence in p is a Morley sequence $(a_i)_{i<\omega}$ over A such that $a_i \models p$ for all i.

An extremely useful concept for the study of stable theories is stationarity:

Definition 2.1.8. A type $p \in S(A)$ is said to be stationary if for any $B \supset A$, there is a unique non-forking extension of p to B. We denote this extension $p|_B$.

This can be thought of as p having a unique canonical extension to any set of parameters. In particular, it is common to consider the *global* non-forking extension $\mathbf{p}$ of a stationary type p, which is defined as its unique extension to a type over $\mathbb{M}$.

This has the useful consequence of yielding a product operation between stationary types:

Definition 2.1.9. Let $p, q \in S(A)$ be two stationary types. The type of (a, b), where $a \models p$, $b \models q$ and $a \mathop{\smile\!\!\!\!\!\!\,|}_A b$ is unique, and denoted $p \otimes q$.

In particular, for any stationary type p, we can form the iterated product of p with itself, it is denoted $\underbrace{p \otimes \cdots \otimes p}_{n \text{ times}} = p^{\otimes n}$

Stationarity is not an automatic property: in algebraically closed fields, for example, it is connected with issues of absolute irreducibility.

Stationary types are part of the foundation of many stability theory techniques, as we will see in this section, and will be used frequently in this dissertation.

Given this, it would be desirable to identify and produce stationary types. Fortunately, this is possible in any stable theory, if one accepts to work in an harmless extension. Namely, we need to work with *imaginaries*.

As this has been exposed in many places, our discussion of imaginaries will be brief. We refer the reader to [38], Section 8.3 for more details. For this section, we will always consider a complete first order theory T.

Imaginaries were introduced by Shelah in [37], with the goal of producing canonical parameters for definable sets.

Definition 2.1.10. A tuple $d \in \mathbb{M}$ is said to be a canonical parameter for a definable set X if $\sigma(X) = X$ if and only if $\sigma(b) = b$, for any σ automorphism of $\mathbb{M}$.

The existence of canonical parameters is not guaranteed. For example, in an infinite model T of the theory of equality, the set $\{a, b\}$, for $a \neq b$, has no canonical parameter.

We invite the reader to contrast this with the situation in an algebraically closed field: if $a \neq b$, then an automorphism fixes $\{a, b\}$ if and only if it fixes the polynomial $(x-a)(x-b)$ if and only if it fixes $\{ab, a+b\}$ pointwise. In fact, symmetric polynomials can be used to show that ACF *eliminates imaginaries*:

Definition 2.1.11. The theory T eliminates imaginaries if any class e/E of an $\emptyset$-definable equivalence relation E has a canonical parameter.

In a theory eliminating imaginaries, every definable set has a canonical parameter, hence the usefulness of this property. But how can we make sure that a given theory eliminates imaginaries? From the previous definition, we see that we need only add canonical parameters for $\emptyset$-definable equivalence relations.

This is done as follows: fix a model $M \models T$. For each $\emptyset$-definable equivalence relation E, add a quotient sort M/E and a quotient map $\pi_E : M \to M/E$. The theory of this (multi-sorted) structure is denoted T^{eq}, and this model is denoted M^{eq}. The original model M is called the home sort, and other sorts are called imaginary sorts. We again refer the reader to [38], Section 8.3 for more details on the construction.

The main properties of this theory are:

Fact 2.1.12. *Let $\mathbb{M}$ be the monster model of T.*

1. *The $\emptyset$-definable sets of the home sort in $\mathbb{M}^{\mathrm{eq}}$ are the same as the ones of $\mathbb{M}$*
2. *The theory T^{eq} eliminates imaginaries*
3. *T is stable if and only if T^{eq} is stable.*
4. *The same goes for λ-categoricity.*

Some important examples of theories eliminating imaginaries are algebraically closed fields, differentially closed fields of characteristic zero and compact complex varieties.

If one is only interested in model-theoretic properties of a theory, there is often no harm in working in T^{eq}, and we will do so in this dissertation. However, bear in mind that for theories that do not have elimination of imaginaries, finding a small, or canonical, family of imaginaries sufficient to eliminate them all can be a non-trivial issue.

Let us now go back to stability, and consider a stable theory T, and its monster model $\mathbb{M}$. We now also have constructed the model $\mathbb{M}^{\mathrm{eq}} \models T^{\mathrm{eq}}$. We will denote $\mathrm{acl}^{\mathrm{eq}}(A)$ the algebraic closure of A in $\mathbb{M}^{\mathrm{eq}}$. The key property of the imaginary algebraic closure is the following:

Lemma 2.1.13. *Let A be any set of parameters. Any type $p \in S(\mathrm{acl}^{\mathrm{eq}}(A))$ over $\mathrm{acl}^{\mathrm{eq}}(A)$ is stationary.*

If A is a set of parameters, and a is some tuple, then we call the *strong type* of a over A, and denote $\mathrm{stp}(a/A)$, the type $\mathrm{tp}(a/\mathrm{acl}^{\mathrm{eq}}(A))$ (which is thus stationary).

It is common, when studying stable theories, to assume elimination of imaginaries. As we have seen, this is mostly harmless if not working with a specific theory. For the remainder of this section, we will assume that T eliminates imaginaries, an assumption often denoted as $T = T^{\mathrm{eq}}$ in the literature. In particular, algebraic closure is always assumed to be the imaginary algebraic closure, so types over algebraically closed sets are always stationary.

One of the defining features of stationary types (assuming elimination of imaginaries) is:

Theorem 2.1.14. *Let $p \in S(A)$ be stationary. Then there is a unique definably closed set $\mathrm{Cb}(p)$, the canonical base of p, satisfying the following:*

1. $\mathrm{Cb}(p) \subset \mathrm{dcl}(A)$
2. *For any $B \subset A$, p does not fork over B if and only if $\mathrm{Cb}(p) \subset \mathrm{acl}(B)$*
3. *Consider the global extension $\mathbf{p}$ of p. If $\sigma \in \mathrm{Aut}(\mathbb{M})$, then σ fixes the type $\mathbf{p}$ if and only if it fixes $\mathrm{Cb}(p)$ pointwise.*

Note that as a consequence of 2., if $C \supset A$, then $\mathrm{Cb}(p|_C) = \mathrm{Cb}(p)$: the canonical base is invariant under taking a non-forking extension. Moreover, the following very useful lemma tells us where to look for this canonical base:

Lemma 2.1.15. *Let $p \in S(A)$ be stationary, and $(a_i)_{i<\omega}$ be a Morley sequence in p. Then* $\mathrm{Cb}(p) \subset \mathrm{dcl}(a_i, i < \omega)$.

Thus, the data needed to define any stationary type is contained in the definable closure of a sequence of its realizations. This is an extremely strong property.

The following theorem summarizes properties of non-forking in stable theories:

Theorem 2.1.16. *Denote $A, B, C, \cdots$ sets of parameters, and $a, b, \cdots$ tuples. If T is stable, non-forking has the following properties:*

- *(Monotonicity and Transitivity) $a \perp_A CB$ if an only if $a \perp_A B$ and $a \perp_{AB} C$.*
- *(Symmetry) $a \perp_A b$ if and only if $b \perp_A a$.*
- *(Finite Character) If $p \in S(B)$ forks over A, then there is a finite $B_0 \subset B$ such that $p|_{AB_0}$ forks over A.*
- *(Local Character) For any a, A, there is $A_0 \subset A$ of cardinality at most $|T|$ such that $a \perp_{A_0} A$.*
- *(Existence) Any type $p \in S(A)$ has a non-forking extension to any set containing A.*
- *(Algebraic Closure):*

  1. *If $p = \mathrm{tp}(b/A)$, where $b \in \mathrm{acl}(A)$, then p does not fork over A.*
  2. *If $a \perp_A a$, then $a \in \mathrm{acl}(A)$.*

- *(Stationarity) If $p \in S(A)$, with A algebraically closed, and $a, b \models p$ with $a \perp_A B$ and $b \perp_A B$, then $\mathrm{tp}(a/AB) = \mathrm{tp}(b/AB)$.*

This theorem allows for algebraic manipulation of the independence symbol, often called "forking calculus". It is a very powerful tool for the study of stable theories, and in concrete examples, non-forking often corresponds to meaningful phenomena (such as transcendence in algebraically closed fields).

Non-forking sometimes comes attached to a notion of dimension, or rank. Two ranks will be of particular interest to us in this dissertation: Morley rank and Lascar rank.

Observe that forking defines a partial ordering on types by $q < p$ if and only if q is a forking extension of p. We can then define the ordinal-valued *foundation rank* of a type inductively:

- For a successor ordinal α, define $\text{rank}(p) \geq \alpha + 1$ if there is $q < p$ such that $\text{rank}(q) \geq \alpha$
- For limit β, set that $\text{rank}(p) \geq \beta$ if $\text{rank}(p) \geq \alpha$ for all $\alpha < \beta$

We can then define $\text{rank}(p) = \alpha$ if $\text{rank}(p) \geq \alpha$ and *not* $\text{rank}(p) \geq \alpha + 1$. We set $\text{rank}(p) = \infty$ if $\text{rank}(p) \geq \alpha$ for all ordinal α.

So by algebraicity, the rank of any algebraic type is set to zero. If the only forking extension of a type is algebraic, it is assigned rank one, and we can proceed inductively from there.

This rank is called the Lascar rank, or U-rank, and is denoted $\text{rank}(p) = U(p)$, for a type p.

Definition 2.1.17. Let T be a stable theory. If every type has ordinal-valued Lascar rank, T is said to be superstable.

This is the only definition of superstability we will use, but there are many others, and we invite the reader to consult [29] for more information.

We can also, given a tuple a and a set of parameters C, define $U(a/A) = U(\text{tp}(a/A))$. In a superstable theory, the Lascar rank completely captures non-forking:

Proposition 2.1.18. *Assume T superstable. Let a be a tuple, and B, C parameters. Then $a \mathop{\smile\!\!\!\!\!\mid}_B C$ is and only if $U(a/B) = U(a/BC)$.*

Finally, a very useful fact is *additivity* of Lascar rank for finite rank types:

Lemma 2.1.19. *Assume that $U(a, b/A) < \omega$. Then $U(a, b/A) = U(a/A) + U(b/aA)$.*

This is no longer valid if the rank is greater than ω, but some weaker formula, involving Cantor normal sum of ordinals, still holds. See [29], Chapter 1, Lemma 3.26, for more details.

The other notion of rank that will come in handy is *Morley rank*. It is more conveniently defined for formulas first, and then for types.

Let $\phi(x)$ be a formula with parameters in the monster model $\mathbb{M}$, we can define its Morley rank inductively by:

- $\mathrm{RM}(\phi \geq 0)$ if ϕ is consistent.
- $\mathrm{RM}(\phi) \geq \alpha+1$ if there is an infinite family $\{\psi_i(x), i < \omega\}$ which all imply $\phi(x)$, are pairwise inconsistent, and with $\mathrm{RM}(\psi_i) \geq \alpha$ for all i.
- If β is a limit ordinal, set $\mathrm{RM}(\phi) \geq \beta$ if $\mathrm{RM}(\phi) \geq \alpha$ for all $\alpha < \beta$.

We can then define Morley rank of a formula as follows:

- $\mathrm{RM}(\phi) = -\infty$ if ϕ is inconsistent.
- $\mathrm{RM}(\phi) = \infty$ if $\mathrm{RM}(\phi) \geq \alpha$ for all ordinal α.
- If there is α such that $\mathrm{RM}(\phi) \geq \alpha$ but *not* $\mathrm{RM}(\alpha) \geq \alpha + 1$, set $\mathrm{RM}(\phi) = \alpha$.

The Morley rank of a type is then defined as $\mathrm{RM}(p) = \min\{\mathrm{RM}(\phi), \phi \in p\}$, and if a is a tuple and C a set of parameters, we define $\mathrm{RM}(a/C) = \mathrm{RM}(\mathrm{tp}(a/C))$.

Definition 2.1.20. A theory T is said to be totally transcendental if every formula has ordinal-valued Morley rank.

In totally transcendental theories, forking is entirely controlled by Morley rank:

Proposition 2.1.21. *Assume T is totally transcendental. Let a be a tuple, and B, C be sets of parameters. Then $a \mathop{\smile\!\!\!\!\!\!\mid}_B C$ if and only if* $\mathrm{RM}(a/B) = \mathrm{RM}(a/BC)$.

Unlike Lascar rank, Morley rank is not additive in general. However, it is for one very important class of formulas:

Definition 2.1.22. A formula $\phi(x)$ is said to be strongly minimal if for any formula $\psi(x)$, either $\phi(\mathbb{M}) \cap \psi(\mathbb{M})$ or $\phi(\mathbb{M}) \cap \neg\psi(\mathbb{M})$ is finite.

A theory T is said to be strongly minimal if $x = x$ is a strongly minimal formula.

Note that it is important to be working in the monster model, as the theory $\mathrm{Th}(\mathbb{N}, <)$ is *not* strongly minimal, but in the model $(\mathbb{N}, <)$, the formula $x = x$ does satisfy the assumptions of the previous definition.

Lemma 2.1.23. *In a strongly minimal theory, Morley rank is additive, i.e. for all tuples a, b and set of parameters C, we have $\mathrm{RM}(a, b/C) = \mathrm{RM}(a/C) + \mathrm{RM}(b/aC)$.*

One might wonder how these two ranks are related. All that holds in general is:

Fact 2.1.24. *For any tuple a and set of parameters A, we have $U(a/A) \leq \mathrm{RM}(a/A)$.*

Equality may not hold. For example, in the theory $\mathrm{Th}(\mathbb{Z}, +)$, there are types with undefined Morley rank but finite Lascar rank. Even in totally transcendental theories, the two ranks might differ, as in differentially closed fields of characteristic zero (although it is non-trivial to prove, see [15]).

Lastly, let us mention a key relationship between two types: orthogonality.

Definition 2.1.25. If $p, q \in S(A)$ are two types, we say that p and q are *weakly orthogonal* if whenever $a \models p$ and $b \models q$, we have $a \mathop{\smile\!\!\!\!\!\!|}_A b$.

If $p \in S(A)$ and $q \in S(B)$ are stationary types, we say that p and q are *orthogonal* if whenever $C \supset A \cup B$, $a \models p|_C$ and $b \models q|_C$, we have $a \mathop{\smile\!\!\!\!\!\!|}_C b$.

Intuitively, two types are non-orthogonal if (possibly after taking their non-forking extension), there is a non-trivial relationship between their realizations. In that light, the following lemma is structurally strong:

Lemma 2.1.26. *Let $p \in S(A)$ be a stationary non algebraic type of finite Lascar rank. There is some stationary type q such that $U(q) = 1$ and p is not orthogonal to q.*

Repeated applications of this lemma will yield much more, as we will see in the next section.

One could wonder, since in a finite Lascar rank theory, any type is non-orthogonal to a rank one type, how many rank one types there are in a given theory. In fact, it is correct to consider the number of rank one types up to non-orthogonality. Of particular interest is:

Definition 2.1.27. The theory T is said to be unidimensional if any two non algebraic stationary types p and q are non-orthogonal.

In that case, there is at most one Lascar rank one type up to non-orthogonality, hence the name unidimensional. In fact, such a rank one type has to exist, because of the following theorem:

Theorem 2.1.28. *Unidimensional theories are superstable.*

A convenient consequence of unidimensionality, that we will use in Chapter 5, is:

Lemma 2.1.29. *If T is unidimensional and totally transcendental, then Morley rank equals Lascar rank. In particular, Morley rank is additive on finite rank tuples.*

We refer to [29], Chapter 1, Section 5, for more details on unidimensional theories.

When a theory fails to be unidimensional, it is desirable to understand and classify its rank one types. An important tool is the combinatorial geometry associated to any rank one type:

Definition 2.1.30. A (combinatorial) pregeometry is a set S equipped with a closure operation $\mathrm{cl} : \mathcal{P}(S) \to \mathcal{P}(S)$, such that for all $X, Y \subset S$ and $a, b \in S$:

- (Idempotence) $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$
- (Monote Increasing) $X \subset \mathrm{cl}(X)$
- (Exchange) If $a \in \mathrm{cl}(X \cup \{b\}) \setminus \mathrm{cl}(X)$, then $b \in \mathrm{cl}(X \cup \{a\})$

- (Finite Character) If $a \in \mathrm{cl}(X)$, then $a \in \mathrm{cl}(Y)$ for some finite $Y \subset X$

A pregeometry (S, cl) is said to be a geometry if $\mathrm{cl}(\emptyset) = \emptyset$ and $\mathrm{cl}(\{a\}) = \{a\}$ for all $a \in S$.

A set X is said to be closed if $X = \mathrm{cl}(X)$.

To any pregeometry (S, cl) we can associate a canonical geometry (S', cl') by letting $S' = \{\mathrm{cl}(\{a\}), a \in S \setminus \mathrm{cl}(\emptyset)\}$ and for any $X \subset S$, defining $\mathrm{cl}'(\{\mathrm{cl}(\{a\}), a \in X\}) = \{\mathrm{cl}(\{b\}), b \in \mathrm{cl}(X)\}$.

Another way to construct pregeometries is to *localize*: if (S, cl) is a pregeometry and $A \subset S$, we can define (S, cl_A) as the pregeometry on S given by $\mathrm{cl}_A(X) = \mathrm{cl}(A \cup X)$ for all $X \subset S$.

The exchange property yields the existence of well-behaved notion of dimension and independence. More precisely, if $A, B \subset S$, we say that A is independent over B if for any $a \in A$, we have $a \notin \mathrm{cl}((A \setminus \{a\}) \cup B)$. We say that A_0 is a basis for A over B if $A \subset \mathrm{cl}(A_0 \cup B)$ and A_0 is independent over B. By exchange, all bases have the same cardinality, which we call the dimension of A over B, and denote $\dim(A/B)$.

Geometries and pregeometries arise frequently in model theory, a classical example being:

Proposition 2.1.31. *Let $p \in S(A)$ be a stationary type of Lascar rank one in some stable theory T. Consider the set $S = p(\mathbb{M})$ of realizations of p in a monster model $\mathbb{M} \models T$. If we let $\mathrm{cl}_S(X) = \mathrm{acl}(X \cup A) \cap S$, for any $X \subset S$, then (S, cl_S) form a pregeometry.*

Note that the same can be done replacing S by the set of realizations, in $\mathbb{M}$, of some strongly minimal formula.

Thus, tools from the theory of pregeometries can be used to classify stationary Lascar rank one types. Among the main dividing lines are:

Definition 2.1.32. A pregeometry (S, cl) is said to be:

- Trivial (or disintegrated) if for every $X \subset S$, we have $\mathrm{cl}(X) = \bigcup_{a \in X} \mathrm{cl}(\{a\})$
- Modular if for any closed sets $X, Y \subset S$, we have that X is independent from Y over $X \cap Y$. Equivalently, that $\dim(X) + \dim(Y) - \dim(X \cap Y) = \dim(X \cup Y)$ for any finite dimensional closed sets X and Y.
- Locally modular if its localization to some $a \in S$ is modular.

A stationary Lascar rank one type p is said to have one of these properties if its associated pregeometry does.

Of course, any such type p is either trivial, locally modular and not trivial, or non-locally modular. This is a very important trichotomy in model theory. Intuitively, trivial types should be "set-like" and not be amenable to geometric tools. Locally modular non-trivial types are expected to be "vector space-like". In fact, from any such type, one can construct a type-definable group (see [29], Chapter 5, Section 1 for more details).

It was long expected that non-locally modular types would be "field-like", and in particular that one should be able to interpret a type definable algebraically closed field in such a type. This was known as Zilber's Trichotomy. It was proven to be false by Hrushovski in [12], where a counterexample was constructed.

However, this trichotomy is still a strong guiding line in model theory, and some important theories do satisfy it, meaning the types in said theory satisfy the trichotomy. Examples include differentially closed fields of characteristic zero, a theory that will be central in this dissertation.

2.2 Internality

The central thread of this dissertation is internality of types, by themselves or in families. In this expository section, we will collect definitions and results that will prove useful to us. Most of this material is covered in chapter 7, section 4 of [29].

We fix the convention, which will be standard for this dissertation, of working in a monster model $\mathbb{M}$ of a stable theory T. We moreover assume that T eliminates imaginaries.

Historically, internality was first considered for definable sets, not for types. Let us recall here what we mean by internal definable sets:

Definition 2.2.1 (for any theory T)**.** Let $\phi(\mathbb{M})$ and $\psi(\mathbb{M})$ be definable sets. The set $\phi(\mathbb{M})$ is said to be $\psi(\mathbb{M})$-internal if for some small set B, we have $\phi(\mathbb{M}) \subset \operatorname{dcl}(\psi(\mathbb{M}), B)$.

In this dissertation, we will mostly work with types. We fix a family $\mathcal{P}$ of partial types, over small subsets of the monster model. We let $\mathcal{P}(\mathbb{M})$ be all the tuples in $\mathbb{M}$ realizing some type in $\mathcal{P}$. We will usually make the small abuse of notation of denoting this by $\mathcal{P}$. No confusion should arise from this, as the context will always make clear if we are referring to tuples or types.

When considering families of partial types over varying subsets of parameters (such as, for example, the family of all Lascar rank one types), we will often need to restrict to those over a fixed set of parameters. Given such a family $\mathcal{P}$, and some set of parameters B, we call a tuple a a realization of $\mathcal{P}$ over B if $\operatorname{tp}(a/B)$ is an extension of some $\Phi \in \mathcal{P}$ which is over B (i.e. has its parameters contained in B). The set of all realizations of $\mathcal{P}$ over B is denoted $\mathcal{P}|_B$.

Let us now begin with the basic definition:

Definition 2.2.2. Let $q \in S(A)$ be a stationary complete type. We say that q is $\mathcal{P}$-internal (resp. almost $\mathcal{P}$-internal), or internal to $\mathcal{P}$, if there are $B \supseteq A$, a tuple a realizing $q|_B$, and $c_1, \cdots, c_n$ realizations of $\mathcal{P}|_B$ such that $a \in \operatorname{dcl}(B, c_1, \cdots, c_n)$ (resp. acl).

Notice once again the crucial feature of internality: we introduced the extra parameters B.

An essential property of the family $\mathcal{P}$, which is needed for any of the relevant theorems to hold, is that of *invariance*.

Definition 2.2.3. Let A be a set of parameters. We say that $\mathcal{P}$ is A-invariant if for every $\Psi \in \mathcal{P}$ and every A-automorphism σ, the partial type $\sigma(\Psi)$ is also in $\mathcal{P}$.

If studying a type $q \in S(A)$, internal to a family of types $\mathcal{P}$, what is needed for most classical results to hold is $\mathcal{P}$ to be A-invariant. In practice, we will always consider one of two situations: either parameters for types in $\mathcal{P}$ are contained in A (we say $\mathcal{P}$ is over A), or $\mathcal{P}$ is the family of *all* U-rank one types. In either case, invariance is immediate.

A canonical choice for the family $\mathcal{P}$ is, as stated in the previous paragraph, the family of all types of U-rank one. The main reason to consider this family is that any type, in a superstable theory, will be constructed as an iterated fibration with $\mathcal{P}$-internal fibers.

Definition 2.2.4. The type $p \in S(A)$ is said to be $\mathcal{P}$-analysable if for any $a \models p$, there are $a_0 = a, a_1, \cdots, a_n$ such that for all i:

- $a_{i+1} \in \mathrm{dcl}(a_i A)$
- $\mathrm{tp}(a_i/a_{i+1}A)$ is $\mathcal{P}$-internal

Theorem 2.2.5. *In a superstable theory of finite rank, any type is $\mathcal{P}$-analysable, where $\mathcal{P}$ is the family of all Lascar rank one types.*

Hence internality is of crucial importance for the study of superstable theories of finite rank.

To prove this, one needs to consider non-orthogonality to families of partial types:

Definition 2.2.6. The stationary type $q \in S(A)$ is said to be weakly orthogonal to the family of types $\mathcal{P}$ if for any realization $a \models q$ and any tuple c of realizations of $\mathcal{P}|_A$, we have $a \mathop{\smile\!\!\!\!\!\!\,|}_A c$.

The stationary type $q \in S(A)$ is said to be orthogonal to the family of types $\mathcal{P}$ if for any set of parameters $B \supset A$, any realization $a \models q|_B$ and any tuple c of realizations of $\mathcal{P}|_B$, we have $a \mathop{\smile\!\!\!\!\!\!|}_B c$.

Lemma 2.1.26 then becomes:

Lemma 2.2.7. *Suppose T is superstable. Any stationary, non algebraic type $p \in S(A)$ of finite Lascar rank is non-orthogonal to the family $\mathcal{P}$ of types of Lascar rank one.*

Using this, and an induction on Lascar rank, one can prove analysability. A key lemma is the following:

Lemma 2.2.8. *If $\operatorname{tp}(a/A)$ is non-orthogonal to an A-invariant family of types $\mathcal{P}$, then there is $b \in \operatorname{acl}(a)$ such that $\operatorname{tp}(b/A)$ is $\mathcal{P}$-internal and $a \not\mathop{\smile\!\!\!\!\!\!|}_A b$.*

The proof of this lemma is of interest, as it illustrates techniques frequently used when working with internality. Thus, we will give a sketch here:

Proof of Lemma 2.2.8. We assume $A = \emptyset$ for convenience. By the non-orthogonality assumption, there is a tuple $c \in \mathcal{P}$ and a tuple d such that:

1. $a \not\mathop{\smile\!\!\!\!\!\!|}_d c$
2. $a \mathop{\smile\!\!\!\!\!\!|} d$

Let $b = \operatorname{Cb}(\operatorname{tp}(cd/\operatorname{acl}(a)))$, then $b \in \operatorname{acl}(a)$. We have $cd \mathop{\smile\!\!\!\!\!\!|}_b \operatorname{acl}(a)$. This implies $a \not\mathop{\smile\!\!\!\!\!\!|} b$, as if not, we would get $a \mathop{\smile\!\!\!\!\!\!|} bcd$, a contradiction. As for internality, notice that $b \in \operatorname{dcl}((c_i d_i)_{i=1 \cdots n})$, a Morley sequence in $\operatorname{tp}(cd/\operatorname{acl}(a))$. As $a \mathop{\smile\!\!\!\!\!\!|} d$, forking calculus yields $a \mathop{\smile\!\!\!\!\!\!|} d_1 \cdots d_n$, and thus $b \mathop{\smile\!\!\!\!\!\!|} d_1 \cdots d_n$, as $b \in \operatorname{acl}(a)$. Moreover, by invariance, all c_i are in $\mathcal{P}$, which yields internality of $\operatorname{tp}(b/\emptyset)$.

□

Let us discuss some basic results regarding internality. Recall that in the definition, we had to introduce extra parameters B. This is the key point, and knowing where to find these parameters is crucial:

Lemma 2.2.9. *Let A be a small set of parameters. Suppose $\mathcal{P}$ is a family of partial types over A, and q is a $\mathcal{P}$-internal stationary type over A. Then there exist a partial A-definable function $f(y_1, \cdots, y_m, z_1, \cdots, z_n)$, a sequence $a_1, \cdots, a_m$ of realizations of q, and a sequence $\Psi_1, \cdots, \Psi_n$ of partial types in $\mathcal{P}$, such that for any a realizing q, there are c_i realizing Ψ_i, for $i = 1 \cdots n$, such that $a = f(a_1, \cdots, a_m, c_1, \cdots, c_n)$.*

Hence internality is witnessed by a definable function, and the extra parameters can be chosen to be a sequence of realizations of q. The tuples $a_1, \cdots, a_m$ of this lemma are called a fundamental system of solutions for q over $\mathcal{P}$.

In fact we will define, for any q:

Definition 2.2.10. Suppose the type q is internal to $\mathcal{P}$. Then a tuple $\overline{a}$ of realizations of q is said to be a fundamental system of solutions for q over $\mathcal{P}$ is for any $b \models q$, we have $b \in \operatorname{dcl}(\overline{a}, \mathcal{P})$.

If some (any) realization of q is a fundamental solution for q, then q is said to be a fundamental type.

By carefully examining the proof of 2.2.9 in [29], we can obtain the following refinement, which will be useful:

Fact 2.2.11. *The $a_1, \cdots, a_m$ of Lemma 2.2.9 can be chosen to be independent realizations of q.*

Hence any $\mathcal{P}$-internal type q has a fundamental system of solutions consisting of independent realizations.

A lot of the structural power of internal types come from the following classical result, due to Hrushovski at this level of generality:

Theorem 2.2.12. *Suppose $q \in S(A)$ is internal to a family of types $\mathcal{P}$ over A, an algebraically closed set of parameters. Then there are an A-type-definable group G and an A-definable group action of G on the set of realizations of q, which is naturally isomorphic (as a group action), to the group $\mathrm{Aut}(q/\mathcal{P}, A)$ of permutations of the set of realizations of q, induced by automorphisms of $\mathbb{M}$ fixing $\mathcal{P} \cup A$ pointwise.*

The group $\mathrm{Aut}(q/\mathcal{P}, A)$ is called the binding group of q over $\mathcal{P}$.

Using this Theorem, one can recover type-definable stable groups from internality, an omnipresent phenomenon in finite rank theories. Stable groups are rather well-behaved objects, and have been studied by model theorists and group theorists alike. The reader is invited to consult [35] for an introduction to this rich subject. Hence, it should be of no surprise that binding groups, arising from the universal phenomenon of internality, are frequently used to prove structural results. Let us cite Hrushovski's proof that unidimensional theories are superstable [10], Pillay's study of imaginaries in pair of algebraically closed fields [31] and Hrushovski, Palacín and Pillay's characterization of the strong canonical base property [16].

By examining the construction of this group in [29], we obtain the following:

Fact 2.2.13. *If q is internal to $\mathcal{P}$, and $r \in S(\emptyset)$ is the type of a fundamental system of solutions for q, then the binding groups $\mathrm{Aut}(q/\mathcal{P})$ and $\mathrm{Aut}(r/\mathcal{P})$ are $\emptyset$-definably isomorphic.*

Our methods will rely heavily on the action of definable automorphism groups. In particular, the following fact, which states that types over $\mathcal{P}$ correspond to orbits under $\mathrm{Aut}(\mathcal{P})$, will be crucial:

Fact 2.2.14. *If $\mathcal{P}$ is a family of partial types, for any two tuples a and b, we have $\mathrm{tp}(a/\mathcal{P}) = \mathrm{tp}(b/\mathcal{P})$ if and only if there is an automorphism of $\mathbb{M}$, fixing $\mathcal{P}$, and taking a to b.*

The reader is invited to consult the proof of Lemma 10.1.5 in [38], their proof easily adapts to yield this fact. Note that if we replace $\mathcal{P}$ by some small set A, this is a basic result. The fact that it holds over $\mathcal{P}$, which is not a small set, is a feature of stable theories, or more generally, of stably embedded sets.

Lastly, Claim II of the proof of Theorem 7.4.8 in [29] implicitly shows the following:

Fact 2.2.15. *For any family of partial types $\mathcal{P}$ over $\emptyset$ and tuple a, we have that* $\mathrm{tp}(a/\operatorname{dcl}(a) \cap \mathcal{P}) \models \mathrm{tp}(a/\mathcal{P})$.

Let us note, before moving on, that the machinery of internality and binding groups is relevant outside of stable theories, using stable embeddedness. A set of (partial) types $\mathcal{P}$ is said to be stably embedded if for any tuple a, we have the conclusion of fact 2.2.15, that is $\mathrm{tp}(a/\operatorname{dcl}(a) \cap \mathcal{P}) \models \mathrm{tp}(a/\mathcal{P})$. We refer the reader to [3] for a nice exposition of stable embeddedness, as well as [13] and [8], where the corresponding theory of binding groups and groupoids was fully developed.

2.3 (Type)-Definable Groupoids

As we move from internality to relative internality, the relevant algebraic objects will become groupoids. For the comfort of our reader, we will now quickly introduce some relevant facts about groupoids. First, the definition:

Definition 2.3.1. A groupoid $\mathcal{G}$ is a non-empty category such that every morphism is invertible.

As a category, the groupoid can then be simply seen as the data of its set of morphisms, its set of objects, along with domain, codomain and (partial) composition. Since every morphism is invertible, there is also an inverse map.

This data is subject to the standard category theory axioms of existence of an identity for each object and associativity of composition, as well as existence of an inverse for each morphism. With all this set up, it is straightforward to write out first order axioms for groupoids.

Groupoids generalize groups. Indeed, every object a of a groupoid $\mathcal{G}$ gives rise to the group $\mathrm{Mor}(a,a)$, called the isotropy group of a. But we also have the extra morphisms $\mathrm{Mor}(a,b)$, for any $a,b \in \mathrm{Ob}(\mathcal{G})$. Remark that a group is then exactly a groupoid with only one object.

The set $\mathrm{Mor}(a,b)$ could be empty if $a \neq b$. This will actually have some meaningful model-theoretic content, and we can define:

Definition 2.3.2. If $\mathcal{G}$ is a groupoid and $a \in \mathrm{Ob}(\mathcal{G})$, then the connected component of a is the set $\{b \in \mathrm{Ob}(\mathcal{G}) : \mathrm{Mor}(a,b) \neq \emptyset\}$. A groupoid is connected if it has only one connected component, and totally disconnected if the connected component of any object is itself.

Since we are interested in definable, or type-definable, objects, we need to define these notions for groupoids.

Definition 2.3.3. A groupoid $\mathcal{G}$ is definable if the sets $\mathrm{Ob}(\mathcal{G})$ and $\mathrm{Mor}(\mathcal{G})$ are definable, and the composition, domain, codomain and inverse maps are definable. It is type-definable is these sets and maps are type-definable.

Remark 2.3.4. An important implication of this definition is that the morphism sets $\mathrm{Mor}(a,b)$ are uniformly definable (resp. type-definable) over the parameters (a,b). Indeed, we have $\mathrm{Mor}(a,b) = \{\sigma \in \mathrm{Mor}(\mathcal{G}), \mathrm{dom}(\sigma) = a, \mathrm{cod}(\sigma) = b\}$, which is an $\{a,b\}$-definable condition.

The behavior of groupoids under model-theoretic assumptions seems to be mostly unexplored so far. They are mostly studied for their relevance to internality, see for example [7] or [13].

2.4 Some Interesting Theories

Some stable theories proved to be rich in examples of internal types. In this dissertation, we will focus mostly on differentially closed fields and, to a lesser extent, compact complex manifolds.

2.4.1 Differentially Closed Fields

In this section, we will collect some results that will prove useful to explore differentially closed fields. Our main reference is [21], Chapter 2, which is a great introduction to differentially closed fields, aimed at model theorists. Any result given without a reference can be found there. We will also refer to [20], which contains a more algebraic development of the Galois theory of Picard-Vessiot extensions.

By convention, all rings will be commutative and with a unit. The first definition is that of a differential ring:

Definition 2.4.1. A differential ring R is a ring equipped with an additive homomorphism $\delta : R \to R$ satisfying Leibniz's rule: for all $xy \in R$, we have $\delta(xy) = \delta(x)y + x\delta(y)$.

From this, it is easy to derive the usual rules for derivatives:

Proposition 2.4.2. *If $x \in (R, \delta)$ and $n \in \mathbb{N}$, then $\delta(x^n) = n\delta(x)x^{n+1}$.*

Proposition 2.4.3. *If $x, y \in (R, \delta)$ and y is a unit in R, then $\delta(\frac{x}{y}) = \frac{\delta(x)y - x\delta(y)}{y^2}$.*

Examples include the trivial derivation $\delta(x) = 0$ for any ring R, or $\mathcal{C}^\infty$ functions on a real interval with the usual derivation.

Just as the right objects to quotient rings are ideals, quotienting differential rings is done via *differential* ideals:

Definition 2.4.4. Let R be a differential ring. An ideal I of R is said to be a differential ideal if for any $a \in I$, we have $\delta(a) \in I$. We can then form the quotient differential ring R/I.

If $A \subset R$, we denote $< A >$ the differential ideal generated by A.

Polynomials rings also carry some natural differential rings structure:

Example 2.4.5. Let R be a ring, we define a differential ring structure on $R[X]$ by setting $\delta(r) = 0$ for all $r \in R$ and $\delta(X) = 1$. This is enough to prescribe the value of δ on any polynomial, as we can apply Leibniz's rule.

We also can define an analogue of polynomial rings, called differential polynomial rings:

Definition 2.4.6. Let (R, δ) be a differential ring, we define $R\{X\}$, the ring of differential polynomials over R, by letting $R\{X\} = R[X_1, X_2, \cdots]$ and extending δ by setting $\delta(X_i) = X_{i+1}$.

We will often denote $\delta(X)$ by X' and $\delta^2(X)$ by X''. Moreover, for any n, we will write $\delta^n(X)$ as $X^{(n)}$.

Definition 2.4.7. If $f \in R\{X\}$, the order of f is the largest n such that $X^{(n)}$ appears in f.

We can define rings of differential polynomials in many variables $K\{X_1, \cdots, X_n\}$ in the exact same fashion.

Of great importance in the study of differential rings are the *constants*:

Definition 2.4.8. If (R, δ) is a differential ring, we denote C_R the kernel of δ, or simply C when there is no ambiguity. It is called the constants of R.

In this dissertation, we will be mainly concerned with differential fields. One can develop a theory for differential fields of positive characteristic, but its study is outside of the scope of this thesis (see [39]). Hence in this chapter, all our fields will be of characteristic zero. Note that if $K \subset L$ are differential fields and $\alpha \in L$, then the set $I = \{f \in K\{X\}, f(\alpha) = 0\}$ is a prime differential ideal of $K\{X\}$.

There are analogues of Hilbert's Basis Theorem and Primary Decomposition Theorem for differential fields. To state them, we will need to define the following:

Definition 2.4.9. Let R be a differential ring and $I \subset R$ be an ideal. We denote $\sqrt{I}$ the smallest *radical* differential ideal containing I (recall that an ideal I is radical if whenever $a^n \in I$ for some n, then $a \in I$).

A radical differential ideal I is finitely generated if there are $a_1, \cdots, a_n \in R$ such that $I = \sqrt{< a_1, \cdots, a_n >}$.

We can now state our two theorems:

Theorem 2.4.10 (Ritt-Raudenbush Basis Theorem)**.** *Let K be a differential field. Then every radical differential ideal of $K\{X_1, \cdots, X_n\}$ is finitely generated.*

This is the analogue of Hilbert's Basis Theorem, which can be used to prove:

Theorem 2.4.11 (Decomposition Theorem)**.** *Let K be a differential field. Any radical differential ideal in $K\{X_1, \cdots, X_n\}$ is the intersection of a finite number of prime differential ideals.*

A proof of the Ritt-Raudenbush Theorem is given in [21]. It requires to dive a bit deeper into differential polynomial rings, and we will skip the proof. However, this theorem, as well as the Decomposition Theorem, will always be in the background of our work.

In studying differential fields, there are many ways to generate rings and fields. We summarize the ones we'll use here:

Notation. Let $k \subset K$ be differential fields and $S \subset K$, we denote:

- $k[S]$ the ring generated by k and S
- $k(S)$ the field generated by k and S
- $k\{S\}$ the differential field generated by k and S

As this dissertation's main subject is model theory, we do need to specify a language to study differential fields, as well as axioms in this language.

Definition 2.4.12. The language $\mathcal{L}_{\mathrm{d-r}}$ of differential rings consists of the language of rings, i.e. binary function symbols $+, -, \cdot$ and constant symbols 0 and 1, together with an extra unary function symbol δ.

The $\mathcal{L}_{\mathrm{d-r}}$-theory of differential rings (resp. fields) is given by the theory of rings (resp. fields) together with axioms stating additivity of δ and Leibniz's rule.

From the point of view of model theory, some of the most well-behaved infinite fields are algebraically closed fields, as their theory is strongly minimal. The differential algebraic analogues are differentially closed fields. Their theory is given by the following axioms:

Definition 2.4.13. The $\mathcal{L}_{\mathrm{d-r}}$ theory DCF_0 of diffentially closed fields of characteristic zero is given by axioms for differential fields, and for any differential polynomials f and g such that the order of g is less than the order of f, the axiom $\exists x f(x) = 0 \wedge g(x) \neq 0$.

It can easily be proved using compactness and rings of differential polynomials that differentially closed fields exist. Here are a few useful facts:

Fact 2.4.14. *The theory DCF_0 of differentially closed fields of characteristic zero:*

- *is complete and model complete*
- *has quantifier elimination*
- *has elimination of imaginaries*
- *is ω-stable of Morley rank ω*

One can prove that DCF_0 is the model companion of the theory of differential fields of characteristic zero. Note that the theory of differential fields of characteristic

p also have a model companion, denoted DCF_p. However, this theory is not as well-behaved, in particular it is not ω-stable (but it is stable).

Let $\mathbb{M} \models \mathrm{DCF}_0$ be a monster model. As DCF_0 is ω-stable, given a tuple $\overline{a} \in \mathrm{DCF}_0$ and a set of parameters B, there are two competing notions for the rank of $\overline{a}$ over B, the Morley rank and Lascar rank of $\mathrm{tp}(\overline{a}/B)$. If the set B is a a field k, there is a third notion, which we will call the dimension of a over k and denote $\dim(a/k)$, given by the transcendence degree of $k\{a\}$, the differential field generated by a, over k.

We will call X a differential algebraic variety if it is the zero-set of a finite collection of differential polynomial equations in $\mathbb{M}$. In particular, such an X is definable and it has a smallest differential field of definition, denoted k. Namely, the set X is defined by polynomials with coefficients in k, and k is the smallest differential field over which such a definition exists. We can then define the dimension of X, denoted $\dim(X)$, as $\max_{a \in X}\{\dim(a/k)\}$.

The structure of strongly minimal sets in DCF_0 turns out to be extremely rich, while still respecting Zilber's trichotomy. There are example of disintegrated, non disintegrated locally modular and non-locally modular strongly minimal set. Moreover, both the locally modular and non-locally modular cases are well understood and classified. However, classifying disintegrated strongly minimal sets is still an ongoing project in model theory.

Recall that if $\mathbb{M} \models \mathrm{DCF}_0$, the constants of $\mathbb{M}$ is a definable set given by the formula $\delta(x) = 0$. We denote it $\mathcal{C}(\mathbb{M})$, or $\mathcal{C}$ when there is no ambiguity. One can easily prove:

Fact 2.4.15. *If $\mathbb{M}$ is a differentially closed field, then its ring of constants $\mathcal{C}(\mathbb{M})$ is an algebraically closed field.*

In the non-locally modular case, it can be shown (see [32]) that any non-locally modular strongly minimal set is non-orthogonal to the constants. Hence, studying internality to non-locally modular strongly minimal sets is equivalent to studying internality to the constants.

A well know class of sets internal to the constants are solution sets of homogeneous linear differential equations. That is, an equation of the form $L(X) = \sum_{i=1}^{n} a_i X^{(i)} = 0$ for some n and $a_i \in \mathbb{M} \models \mathrm{DCF}_0$. Using some linear algebra, one can prove:

Theorem 2.4.16. *Let $\mathbb{M} \models \mathrm{DCF}_0$ and $L(X) = 0$ be a homogeneous linear differential equation. Then there are $x_1, \cdots, x_n \in \mathbb{M}$, solutions of $L(X) = 0$, linearly independent over $\mathcal{C}(\mathbb{M})$, such that the solution set for $L(X) = 0$ in $\mathbb{M}$ is exactly the linear span of $x_1, \cdots, x_n$ over $\mathcal{C}(\mathbb{M})$.*

These equations are the start of a very rich Galois theory, based on Picard-Vessiot extensions:

Definition 2.4.17. Let l/k be differential fields. We say that l is a Picard-Vessiot extension of k if there is a homogeneous linear differential equation $L(X) = 0$ and $x_1, \cdots, x_n$ a fundamental system of solutions such that $l = k\{x_1, \cdots, x_n\}$ and $\mathcal{C}_k = \mathcal{C}_l$. We define the order of l/k as the order of the linear differential polynomial L.

The following can be found in [20]:

Theorem 2.4.18. *Let l/k be a Picard-Vessiot extension of order n, and $G(l/k)$ be the group of differential field automorphisms of l fixing k pointwise. Then $G(l/k)$ is isomorphic to an algebraic subgroup of $\mathrm{Gl}_n(C_k)$*

From a model-theoretic perspective, note that any homogeneous linear differential equation $L(X) = 0$ gives rise, in $\mathbb{M} \models \mathrm{DCF}_0$, to a Picard-Vessiot extension. Indeed, one can consider the fields $k = \mathcal{C}(\mathbb{M})\{a_1, \cdots, a_n\}$ and $l = k\{x_1, \cdots, x_n\}$ (keeping the same notations). Then l/k is a Picard-Vessiot extension. Moreover, the set $L(X) = 0$ is definable and $\mathcal{C}$-internal. Its binding group is the same as the Galois group of Theorem 2.4.18.

For a thorough introduction to differential Galois theory, we refer the reader to [20].

In this dissertation we will often go one step further and study types analysable over the constants. A systematic way to produce analysable types from an internal one is to consider its preimages under either the derivative of the logarithmic derivative. Recall that the logarithmic derivative is defined as $\delta\log(x) = \frac{\delta(x)}{x}$ for any $x \in \mathbb{M}$.

We have the following easy fact:

Fact 2.4.19. *Let $a \in \mathbb{M}$, a differentially closed field. Then the sets $\delta^{-1}(a)$ and $\delta\log^{-1}(a)$ are $\mathcal{C}$ internal. Moreover, the binding group of $\delta^{-1}(a)$ is a definable subgroup of $\mathbb{G}_a(\mathcal{C})$ (thus either the whole group or trivial), and the binding group of $\delta\log^{-1}(a)$ is a definable subgroup of $\mathbb{G}_m(\mathcal{C})$ (thus either the whole group or a finite cyclic group).*

Therefore, again fixing $\mathbb{M} \models \mathrm{DCF}_0$ and p a $\mathcal{C}$-internal type, we have that any type extending $\delta^{-1}(p) = \{\alpha \in \mathbb{M}, \delta(\alpha) \models p\}$ or $\delta\log^{-1}(p) = \{\alpha \in \mathbb{M}, \delta\log(\alpha) \models p\}$ is 2-analysable over the constants.

Note that in some cases, these types are actually internal to the constants. For example, if one consider q to be the generic type of the constants, then $\delta^{-1}(q)$ is the generic type of the differential-algebraic variety given by $x'' = 0$, which is $\mathcal{C}$-internal as the solution set of a linear differential equation. By contrast, one can prove that $\delta\log^{-1}(q)$ is not even almost $\mathcal{C}$-internal. See [4] for a pleasant proof of that fact.

The question of when exactly are the types $\delta^{-1}(q)$ and $\delta\log^{-1}(q)$ internal does not have a general solution, but much progress has been made recently. We invite the reader to consult [17] for a systematic construction of strictly 2-analysable types of the form $\delta\log^{-1}(q)$, and [18] for a partial characterization if q is of Morley rank one.

To prove that types obtained by such preimage constructions are not internal, we will usually have to use one of the following:

Lemma 2.4.20 ([20], Remark 1.11.1)**.** *If $E = F(z)$ is a differential field extension with $\delta \log(z) \in F$, then either z is transcendental over F or $z^n \in F$ for some $n > 0$.*

and:

Lemma 2.4.21. *If $E = F(z)$ is a differential field extension with $\delta(z) \in F$, and $\mathcal{C}_E = \mathcal{C}_F$, then either z is transcendental over F or $z \in F$.*

Proof. If z is *not* transcendental over F, then there is a polynomial $P(X) = X^n + \sum_{i=0}^{n-1} a_i X^i \in F[X]$ such that $P(z) = 0$. We can pick P to be the minimal polynomial of z over F.

Because $\delta(z) \in F$, we have that $\deg(\delta(P(z))) = n-1$, and its dominant coefficient is $\delta(a_{n-1}) + n\delta(z)$. By minimality of P, we must have $\delta(a_{n-1}) + n\delta(z) = 0$, and this yields that $z = ca_{n-1}$, for some $c \in \mathcal{C}_F$. Hence $z \in F$.

□

2.4.2 Compact Complex Manifolds

In this section, we introduce some basic model theory of compact complex manifolds. As we will only spend limited time in the realm of complex geometry, our introduction will be somewhat succinct, and drawn from Section 2 of [22]. We invite the reader to consult [22] for a more thorough presentation, [23] for a survey of the model theory of compact complex manifolds, as well as [24] for material related to Chapter 4, Section 4.4. We will, for the most part, repeat the material contained in these papers, and all results, unless explicitly mentioned, are from there.

The model theory of compact complex manifolds was first considered by Zilber in [40], where it is proved that they are amenable to model-theoretic methods. More precisely, the structure given by one compact complex manifold, considered in a natural language, is ω-stable of finite Morley rank.

Recall that any of these variety is equipped with the (analytic) Zariski topology, which closed sets are locally given by the vanishing of holomorphic functions. This is a noetherian topology, coarser than the euclidian topology.

From this Zariski topology, we can construct a first order theory, called CCM. We will consider the structure $\mathcal{A}$ equipped with a sort for each compact complex variety. The language $\mathcal{L}$ consists of a predicate for each analytic subset of any product of sort (thus any holomorphic function, for example). By itself, each sort has quantifier elimination and is of finite Morley rank. Moreover, CCM eliminates imaginaries.

We will work in $\mathbb{A}$, a monster model of $\mathrm{Th}(\mathcal{A})$. Note that every point $a \in \mathcal{A}$, as a compact complex variety, is a constant in our language, thus $\mathcal{A} \subset \mathrm{dcl}(\emptyset) \subset \mathbb{A}$. Considering such a non-standard model has no equivalent in complex analytic geometry, and we need to make sense of what these new elements correspond to. As we are model-theorists, this amounts to describing types $\mathrm{tp}(a/b)$, where $a, b \in \mathbb{A}$.

Just as in algebraic geometry, the notion of genericity will play an essential role, allowing us to give concrete meaning to types as generic points:

Definition 2.4.22. Let X be an irreducible Zariski closed set from $\mathcal{A}$. The generic type of X over $\mathcal{A}$ is defined by:

$$p(x) = \{x \in X\} \cup \{x \notin F, F \text{ proper Zariski closed subset of } X\}$$

A generic point of X over $\mathcal{A}$ is a realization of p.

By quantifier elimination and noetherianity of the Zariski topology, this is indeed a complete, consistent type. Note that since any point of $\mathcal{A}$ is in our language, types over $\mathcal{A}$ are really just types over the empty set.

By saturation of $\mathbb{A}$, every Zariski closed set of $\mathcal{A}$ has a generic point. Moreover, if we pick any $a \in \mathbb{A}$, the type $p = \mathrm{tp}(a/\emptyset)$ is the generic point of some Zariski closed set X in $\mathcal{A}$, again by noetherianity. We call X the *locus* of a (and of p). Thus, types of non-standard points over $\emptyset$ simply correspond to generic points of Zariski closed sets.

We only have definable sets and types over non-standard parameters left to describe. By quantifier elimination, every definable set in $\mathbb{A}$ is of the form:

$$Z(\mathbb{A})_s = \{y \in \mathbb{A} : (s, y) \in Z(\mathbb{A})\}$$

where X and Y are sorts of $\mathcal{A}$, $Z \subset X \times Y$ is definable, and $s \in X(\mathbb{A})$. If Z can be chosen Zariski closed, then we say that $Z(\mathbb{A})_s$ is a non-standard Zariski closed set. Again by quantifier elimination, any definable set in $\mathbb{A}$ can be written as a boolean combination of non-standard Zariski closed sets.

So any non-standard Zariski closed set can be seen as a fiber. Moreover, it can be though of as the *generic* fiber of some standard Zariski closed set. More precisely, let $\Gamma = Z(\mathbb{A})_s$ be as above, and let $S \subset X$ be the locus of s. If we consider $T = Z \cap (S \times Y)$, we obtain Γ as a generic fiber of the family of Zariski closed sets:

$$S \times Y \supset T \longrightarrow S$$

because it is the fiber over s, which is generic in S.

Moreover, it can be shown that this characterization is unique in the following sense: if Γ is also given as the generic fiber of some other $R \subset S \times Y$, there there is a Zariski open set $U \subset S$ such that for any $u \in S$, we have $T_u = R_u$.

Using this characterisation, we can define genericity and locus:

Definition 2.4.23. Let B be a set of parameters, and Γ a non-standard Zariski closed set. The set Γ is said to be over B if it is definable with parameters from the set B. If Γ is over B, is is said to be irreducible if it is not the union of two proper non-standard Zariski closed sets over B. Finally, it is said to be absolutely irreducible if it is irreducible over any set of parameters containing B.

Definition 2.4.24. Suppose Γ is an irreducible non-standard Zariski closed set over B. The generic type $p(x)$ of Γ is defined as:

$$p(x) = \{x \in \Gamma\} \cup \{x \notin \Sigma, \Sigma \text{ proper nonstandard Zariski closed subset of } \Gamma \text{ over } B\}$$

and a generic point of Γ over B is a realization of $p(x)$.

If $c \in \mathbb{A}$, then there is a smallest non-standard Zariski closed set over B containing c. We call it the *locus* of c over B.

Much of the machinery of complex analytic geometry has equivalent for non-standard Zariski closed sets. First, for any compact complex variety X, the nonstandard Zariski closed subsets of $X(\mathbb{A})$ are the closed sets of a noetherian topology on $X(\mathbb{A})$.

There is also a dimension, extending the classical one:

Definition 2.4.25. Suppose the non-standard Zariski closed set is obtained as the generic fiber of the holomorphic map $Z \to S$. The dimension of Z is defined as the dimension of the general fibers of $Z \to S$.

If $a \in \mathbb{A}$, and B is a set of parameters, we can thus define $\dim(a/B)$ as the dimension of the locus of a over B.

Note that for this to be well-defined, all these fibers must have the same dimension, which is a non-trivial fact. We invite the reader to consult [5], Chapter 3, for a proof of this.

This dimension is related to forking by:

Theorem 2.4.26. *Suppose a and b are tuples and B is a set of parameters. Then a is independent from b over B is* $\dim(a/B) = \dim(a/bB)$.

Finally, let us state how stationarity and irreducibility are related:

Proposition 2.4.27. *Let* Γ *be the locus of* a *over* B. *Then* $\mathrm{tp}(a/B)$ *is stationary if and only if* Γ *is absolutely irreducible.*

CHAPTER 3

GENERALIZING THE BINDING GROUP

3.1 Introduction and Set Up

All through this chapter, we will fix a stable theory T, eliminating imaginaries, as well as a monster model $\mathbb{M}$ of said theory. Recall once again the following classic result from stability theory:

Theorem 3.1.1. *Suppose $q \in S(A)$ is stationary and internal to a family of types $\mathcal{P}$ over A, an algebraically closed set of parameters. Then there are an A-type-definable group G and an A-definable group action of G on the set of realizations of q, which is naturally isomorphic (as a group action), to the group $\mathrm{Aut}(q/\mathcal{P}, A)$ of permutations of the set of realizations of q, induced by automorphisms of $\mathbb{M}$ fixing $\mathcal{P}(\mathbb{M}) \cup A$ pointwise.*

This theorem is part of a large family of model theory results constructing an algebraic object from an abstract configuration (another example being Hrushovski's group configuration theorem [11]). The goal of this chapter is to expand this theorem to a slightly different setup, hereby adding another member to this family.

The group $\mathrm{Aut}(q/\mathcal{P})$ is the binding group of q over $\mathcal{P}$. Its existence, in a more restrictive context, was first proved by Zilber in [42]. The previous theorem, at this level of generality, was proved by Hrushovski in [10]. A proof is also given in [29], which is the one we will follow to generalize 3.1.1.

Instead of working with internal types, we will use, in this chapter, what we call *relatively internal* types. For the rest of this section, we fix an algebraically closed set of parameters A, a type $q \in S(A)$, and a family of partial types $\mathcal{P}$ over A.

Definition 3.1.2. Let q be a complete type over A, and π an A-definable partial map, defined on any realization of q. The pair (q, π) is said to be relatively $\mathcal{P}$-internal if for any (some) $a \models q$, the type $\mathrm{tp}(a/\pi(a)A)$ is stationary and $\mathcal{P}$-internal.

Note that if $a \models q$, the set $\{x \models q, \pi(x) = \pi(a)\}$ is the set of realizations of a complete type, denoted $q_{\pi(a)}$ in the rest of this chapter.

This definition might seem arbitrary at first, but relative internality does appear regularly in the literature, albeit not in a formalized way. The two main sources of relatively internal pairs are:

- Internal types: if $\mathrm{tp}(a/b)$ is $\mathcal{P}$-internal, then $(\mathrm{tp}(ab), \pi)$ is relatively internal, with π the projection on the b-coordinate.
- Analysability: if $\mathrm{tp}(a/\emptyset)$ is $\mathcal{P}$-analysable, then there is $b \in \mathrm{dcl}(a)$ such that $\mathrm{tp}(a/b)$ is $\mathcal{P}$-internal. Thus $b = \pi(a)$ for some $\emptyset$-definable function π, and $(\mathrm{tp}(a), \pi)$ is relatively $\mathcal{P}$-internal.

In the presence of a relatively internal pair, Theorem 2.2.12 equips us with a type definable binding group $\mathrm{Aut}(q_{\pi(a)}/\mathcal{P}, A)$ for each $\pi(a)$, acting on the fiber $q_{\pi(a)}(\mathbb{M})$. However, this is not the whole story: there is in fact a groupoid action here.

Observe that rather than considering only automorphisms of fibers, we could consider any partial automorphism between any two fibers. More specifically, let us define a groupoid $\mathcal{G}(q, \pi/\mathcal{P}, A)$ as follows:

- its set of objects is the type-definable set $\pi(q)(\mathbb{M})$
- for any two $\pi(a), \pi(b) \in \mathrm{Ob}(\mathcal{G}(q, \pi, \mathcal{P}, A))$, let $\mathrm{Mor}(\pi(a), \pi(b))$ consists of bijections from $q_{\pi(a)}(\mathbb{M})$ to $q_{\pi(b)}(\mathbb{M})$, induced by automorphisms of $\mathbb{M}$ fixing $\mathcal{P} \cup A$ pointwise.

Note that for each $\pi(a)$, this means that $\mathrm{Mor}(\pi(a), \pi(a)) = \mathrm{Aut}(q_{\pi(a)}/\mathcal{P}, A)$. The groupoid action is defined by letting $X = \{(\sigma, a) \in \mathrm{Mor}(\mathcal{G}) \times q(\mathbb{M}) : \mathrm{dom}(\sigma) = \pi(a)\}$, and letting the action map be:

$$X \to q(\mathbb{M})$$

$$(\sigma, a) \to \sigma(a)$$

We can now state the main result of this chapter:

Theorem 3.1.3. *The groupoid $\mathcal{G}$ is isomorphic to an A-type-definable groupoid, and its natural groupoid action on realizations of q is A-definable.*

Hence the binding groups $\mathrm{Aut}(q_{\pi(a)})$ are uniformly type-definable, and fit together in a type-definable groupoid.

3.2 Construction of the Relative Binding Groupoid

Recall our working assumptions: T is a stable theory eliminating imaginaries, and $\mathbb{M}$ is a monster model of T. Theorem 3.1.3 is stated over any algebraically closed set A, but we will here assume, without loss of generality, that we are working over $\emptyset$, and that $\emptyset = \mathrm{acl}(\emptyset)$.

Hence we fix a family $\mathcal{P}$ of partial types over $\emptyset$, as well as $q \in S(\emptyset)$, and an $\emptyset$-definable map π such that (q, π) is relatively $\mathcal{P}$ internal. So for all $a \models q$, the type $q_{\pi(a)} = \mathrm{tp}(a/\pi(a))$ is stationary and $\mathcal{P}$-internal.

For any $a \models q$, the set $q(x) \cup \{\pi(x) = \pi(a)\}$ is the set of realizations of a complete type, denoted $q_{\pi(a)}$. To ease notation, if $\overline{a}$ is a tuple of realizations of q with same image under π, we will denote $\pi(\overline{a})$ their common image.

Recall that there is a groupoid $\mathcal{G}$, whose objects are given by $\pi(q)(\mathbb{M})$, and morphisms $\mathrm{Mor}(\pi(a), \pi(b))$ by the set of bijections from $q_{\pi(a)}(\mathbb{M})$ to $q_{\pi(b)}(\mathbb{M})$, induced by automorphisms of $\mathbb{M}$ fixing $\mathcal{P}$ pointwise, and taking $\pi(a)$ to $\pi(b)$. Our goal is to prove that this groupoid, as well as its action on realizations of q are $\emptyset$-type-definable. We now start the proof, which follows closely the proof of Theorem 7.4.8 from [29]:

Proof of Theorem 3.1.3. First note that the objects are the $\emptyset$-type-definable set $\pi(q)$. So what we have to show is that the set of morphisms is $\emptyset$-type-definable, as well as domain and codomain maps, and composition.

Note that since each π-fiber is $\mathcal{P}$-internal, we can apply Lemma 2.2.9 to any of them, so each type $q_{\pi(a)}$ has a fundamental system of solution. The first step of the proof is to show that these fundamental systems can be chosen uniformly, in the following sense:

Claim 3.2.1. *There exist $r \in S(\emptyset)$, a partial $\emptyset$-definable function $f(y, z_1, \cdots, z_n)$, a sequence $\Psi_1, \cdots, \Psi_n$ of partial types in $\mathcal{P}$. These satisfy that for each $\pi(a) \models \pi(q)$, there is $\overline{a} \models r$ such that $\pi(a) = \pi(\overline{a})$, and for any other $a' \models q_{\pi(a)}$, there are c_i realizing Ψ_i, for $i = 1 \cdots n$, with $a' = f(\overline{a}, c_1, \cdots, c_n)$.*

Proof. Let $\pi(a)$ be a realization of $\pi(q)$. Applying Lemma 2.2.9 to $q_{\pi(a)}$ yields a partial $\pi(a)$-definable function $f(y_1, \cdots, y_m, z_1, \cdots, z_n)$, a sequence $a_1, \cdots, a_m$ of realizations of $q_{\pi(a)}$, and a sequence $\Psi_1, \cdots, \Psi_n$ of partial types in $\mathcal{P}$, such that $q_{\pi(a)} \subset f(\overline{a}, \Psi_1(\mathbb{M}), \cdots, \Psi_n(\mathbb{M}))$.

Denote $\overline{a} = (a_1, \cdots, a_m)$, and $r = \text{tp}(\overline{a}/\emptyset)$. Remark that since $\pi(\overline{a}) = \pi(a) \in \text{dcl}(\overline{a})$, the function f is actually $\emptyset$-definable. By invariance, we see that f, r and $\Psi_1, \cdots, \Psi_n$ satisfy the required properties.

$\square$

We will now fix r, f be as in Claim 3.2.1, and $\Phi(\overline{x}) = \Psi(x_1) \cup \cdots \cup \Psi(x_n)$. Fix $\pi(a)$, $\pi(b)$ and a realization $\overline{a}$ of r in $\pi^{-1}(\pi(a))$. Consider the set $X = \{(\overline{a}, \overline{b}) : \text{tp}(\overline{a}) = \text{tp}(\overline{b}) = r, \text{tp}(\overline{a}/\mathcal{P}) = \text{tp}(\overline{b}/\mathcal{P})\}$, it is the set we will use to encode morphisms. We have:

Claim 3.2.2. *The set X is $\emptyset$-type-definable.*

Proof. Fact 2.2.15 yields that $\text{tp}(\overline{a}/\operatorname{dcl}(\overline{a}) \cap \mathcal{P}) \models \text{tp}(\overline{a}/\mathcal{P})$. Consider the set $\{\lambda_i(\overline{x}) : i \in I\}$ of partial $\emptyset$-definable functions defined at $\overline{a}$ with values in $\mathcal{P}$ (and these are the same at every realization of r). Then $\text{tp}(\overline{a}/\mathcal{P}) = \text{tp}(\overline{b}/\mathcal{P})$ if and only if $\lambda_i(\overline{a}) = \lambda_i(\overline{b})$ for all $i \in I$. Therefore $X = \{(\overline{a}, \overline{b}) : \text{tp}(\overline{a}) = \text{tp}(\overline{b}) = r, \lambda_i(\overline{a}) = \lambda_i(\overline{b}) \text{ for all } i \in I\}$, which is an $\emptyset$-type-definable set. □

Let $r_{\overline{a}} = \text{tp}(\overline{a}/\mathcal{P})$. We then have the following:

Claim 3.2.3. *The map from* $\text{Mor}(\pi(a), \pi(b))$ *to* $r_{\overline{a}}(\mathbb{M}) \cap \{\overline{x} : \pi(\overline{x}) = \pi(b)\}$ *taking* σ *to* $\sigma(\overline{a})$ *is a bijection.*

Proof. First injectivity: suppose $\sigma(\overline{a}) = \tau(\overline{a})$. Every element of $\pi^{-1}(\pi(a)) \cap q(\mathbb{M})$ is written as $f(\overline{a}, c)$, for some $c \models \Phi$, and $\sigma(f(\overline{a}, c)) = f(\sigma(\overline{a}), \sigma(c)) = f(\tau(a), c) = \tau(f(\overline{a}, c))$, so $\tau = \sigma$.

For surjectivity, given $\overline{b} \models r_{\overline{a}}$, since $\overline{a}$ and $\overline{b}$ have the same type over $\mathcal{P}$, by Fact 2.2.14, there is an automorphism of the monster model, fixing $\mathcal{P}$, and taking $\overline{a}$ to $\overline{b}$. The restriction of this automorphism to $q_{\pi(a)}(\mathbb{M})$ belongs to $\text{Mor}(\pi(a), \pi(b))$.

□

By Claim 3.2.3, for any $(\overline{a}, \overline{b}) \in X$, there is a unique $\sigma \in \text{Mor}(\pi(\overline{a}), \pi(\overline{b}))$ such that $\sigma(\overline{a}) = \overline{b}$. And for any $\overline{a} \models r$ and $\sigma \in \text{Mor}(\pi(\overline{a}), \pi(\overline{b}))$, we also have $(\overline{a}, \sigma(\overline{a})) \in X$. However, this correspondence may not be injective: for any $\sigma \in \text{Mor}(\pi(a), \pi(b))$, there are multiple elements of X corresponding to it. We will solve this problem with an equivalence relation.

Claim 3.2.4. *There is a formula* $\psi(\overline{x}_1, \overline{x}_2, y, z)$ *such that for any* $\sigma \in \text{Mor}(\mathcal{G})$, *any* $\overline{a} \models r$, *any* $a \models q$ *such that* $\text{dom}(\sigma) = \pi(a) = \pi(\overline{a})$ *and any* b, *we have* $\models \psi(\overline{a}, \sigma(\overline{a}), a, b)$ *if and only if* $b = \sigma(a)$.

Proof. By the proofs of Claim 3.2.2 and Claim 3.2.1, if $\overline{a}, \overline{b}$ realise r, with $\lambda_i(\overline{a}) = \lambda_i(\overline{b})$ for all i, and c_1, c_2 realise the partial type Φ of Claim 3.2.1, then $f(\overline{a}, c_1) = f(\overline{a}, c_2)$ if and only if $f(\overline{b}, c_1) = f(\overline{b}, c_2)$ (and these are well defined). By compactness, there is a formula $\theta(w)$ and a finite subset $J \subset I$ such that the previous property is true replacing Φ by θ and I by J.

Let the formula $\psi(\overline{x}_1, \overline{x}_2, y, z)$ be $\exists w(f(\overline{x}_1, w) = y \wedge f(\overline{x}_2, w) = z \wedge \theta(w))$. We now check that this formula works. Suppose first that $\overline{a}, \sigma(\overline{a}), a, b$ satisfy it. Then there is $c \models \theta(w)$ such that $f(\overline{a}, c) = a$ and $f(\sigma(\overline{a}), c) = b$.

But as $a \models q$, there is also $d \models \Phi$ such that $f(\overline{a}, d) = a$. Since $d \models \Phi$, it is a realization of $\mathcal{P}$, hence $\sigma(a) = \sigma(f(\overline{a}, d)) = f(\sigma(\overline{a}), d)$.

So $f(\overline{a}, c) = a = f(\overline{a}, d)$, the tuple d is a realization of Φ, and $f(\sigma(\overline{a}, d)) = \sigma(a)$. By choice of ψ, this implies $b = f(\sigma(\overline{a}), c) = f(\sigma(\overline{a}), d) = \sigma(a)$.

Conversely, suppose that $b = \sigma(a)$. Since $\overline{a} \models r$ and $\mathrm{dom}(\sigma) = \pi(a) = \pi(\overline{a})$, there is $c \models \Phi$ such that $f(\overline{a}, c) = a$. Therefore $b = \sigma(f(\overline{a}, c)) = f(\sigma(\overline{a}), c)$, so we can take c to be the w of the formula.

□

Remark that the content of this proposition really is that X 'acts' on q definably, witnessed by the formula ψ. In what follows, we will identify elements of X if their action coincides, which will yield definability of the groupoid.

More precisely, define an equivalence relation E on X as $(\overline{a}_1, \overline{b}_1)E(\overline{a}_2, \overline{b}_2)$ if and only if $\pi(\overline{a}_1) = \pi(\overline{a}_2)$ and for some $\sigma \in \mathrm{Mor}(\mathcal{G})$, we have $\sigma(\overline{a}_1) = \overline{b}_1$ and $\sigma(\overline{a}_2) = \overline{b}_2$. So $(\overline{a}_1, \overline{b}_1)E(\overline{a}_2, \overline{b}_2)$ if and only if the tuples $(\overline{a}_1, \overline{b}_1)$ and $(\overline{a}_2, \overline{b}_2)$ represent the same morphism σ. Then the following is true:

Claim 3.2.5. *E is relatively $\emptyset$-definable on $X \times X$.*

Proof. Recall that we denote $\mathrm{tp}(a/\pi(a))$, for $a \models q$, by $q_{\pi(a)}$. We also denote $r_{\pi(a)} = \mathrm{tp}(\overline{a}/\pi(\overline{a}))$, for some $\overline{a} \models r$ with $\pi(\overline{a}) = \pi(a)$.

We first show that $(\overline{a}_1, \overline{b}_1)E(\overline{a}_2, \overline{b}_2)$ if and only if $\pi(\overline{a}_1) = \pi(\overline{a}_2)$ and for any $a \models q_{\pi(\overline{a}_1)}|_{\overline{a}_1, \overline{a}_2, \overline{b}_1, \overline{b}_2}$ we have $\forall z \psi(\overline{a}_1, \overline{b}_1, a, z) \leftrightarrow \psi(\overline{a}_2, \overline{b}_2, a, z)$.

The left to right direction is immediate. So assume that the right-hand condition holds. There are $\sigma, \tau \in \mathrm{Mor}(\mathcal{G})$ with $\sigma(\overline{a}_1) = \overline{b}_1$ and $\tau(\overline{a}_2) = \overline{b}_2$. Let $\overline{a}_3 \models r_{\pi(\overline{a}_1)}$ be independent from $\overline{a}_1, \overline{a}_2, \overline{b}_1, \overline{b}_2$. If we let $\overline{a}_3 = (a_{3,1}, \cdots, a_{3,n})$, then by independence, we have $\forall z \psi(\overline{a}_1, \overline{b}_1, a_{3,i}, z) \leftrightarrow \psi(\overline{a}_2, \overline{b}_2, a_{3,i}, z)$, for all $1 \leq i \leq n$. Hence $\sigma(\overline{a}_3) = \tau(\overline{a}_3)$. Let a' be any realization of $q_{\pi(\overline{a}_1)}$. Then $a' = f(\overline{a}_3, c)$ for some c, since $\overline{a}_3$ is a realization of $r_{\pi(\overline{a}_1)}$. So $\sigma(a') = \sigma(f(\overline{a}_3, c)) = f(\sigma(\overline{a}_3), c) = f(\tau(\overline{a}_3), c) = \tau(a')$. This is true for any realization a' of $q_{\pi(\overline{a}_1)}$, so $\tau = \sigma$.

Notice that the right-hand condition is equivalent to a formula over $\pi(\overline{a}_1)$ because the stationary type $q_{\pi(\overline{a}_1)}$ is definable over $\pi(\overline{a}_1)$. So if we fix $\pi(a) \models \pi(q)$, there is a formula $\theta_{\pi(a)}(z_1, t_1, z_2, t_2, y)$ over $\emptyset$ such that for any $\overline{a}_1, \overline{a}_2 \models r_{\pi(a)}$, and any $\overline{b}_1, \overline{b}_2$, we have $(\overline{a}_1, \overline{b}_1) E (\overline{a}_2, \overline{b}_2)$ if and only if $\theta_{\pi(a)}(\overline{a}_1, \overline{b}_1, \overline{a}_2, \overline{b}_2, \pi(a))$. A priori, this formula $\theta_{\pi(a)}$ depends on $\pi(a)$, hence we cannot yet conclude that E is relatively definable, let alone relatively $\emptyset$-definable. However, if we can prove that for any $a, b \models q$ the formulas $\theta_{\pi(a)}(z_1, t_1, z_2, t_2, \pi(a))$ and $\theta_{\pi(b)}(z_1, t_1, z_2, t_2, \pi(a))$ are equivalent, we would get relative $\emptyset$-definability.

Note that the formula $\theta_{\pi(a)}$ we obtained is a defining scheme for a formula in the stationary type $q_{\pi(a)} = \mathrm{tp}(a/\pi(a))$. We will use this to show the desired equivalence.

Let $\pi(a), \pi(b) \models \pi(q)$, and σ an automorphism such that $\sigma(\pi(a)) = \pi(b)$. Let $\phi(x, y, z)$ be a formula over $\emptyset$. Since $q_{\pi(a)}$ and $q_{\pi(b)}$ are definable and stationary, there are defining schemes $\theta_{\pi(a)}(z, \pi(a))$ (respectively $\theta_{\pi(b)}(z, \pi(b))$) for $\phi(x, y, z)$ and $q_{\pi(a)}$ (respectively $q_{\pi(b)}$), and the formulas $\theta_{\pi(-)}(z, y)$ are over the empty set. Now, let $\overline{c}$ be a tuple, and a' a realization of $q_{\pi(a)}|_{\overline{c}}$, the unique non-forking extension of $q_{\pi(a)}$ to $\{\pi(a), \overline{c}\}$. Then:

$$
\begin{aligned}
\theta_{\pi(a)}(\overline{c},\pi(a)) &\Leftrightarrow \phi(x,\pi(a),\overline{c}) \in q_{\pi(a)}|_{\overline{c}} \\
&\Leftrightarrow \models \phi(a',\pi(a),\overline{c}) \\
&\Leftrightarrow \models \phi(\sigma(a'),\pi(b),\sigma(\overline{c})) \\
&\Leftrightarrow \phi(x,\pi(b),\sigma(\overline{c})) \in q_{\pi(b)}|_{\sigma(\overline{c})} \text{ because } \sigma(a') \models q_{\pi(b)}|_{\sigma(\overline{c})} \\
&\Leftrightarrow \theta_{\pi(b)}(\sigma(\overline{c}),\pi(b)) \\
&\Leftrightarrow \theta_{\pi(b)}(\overline{c},\pi(a))
\end{aligned}
$$

Applying this to the formula $\forall w \psi(z_1,t_1,x,w) \leftrightarrow (\psi(z_2,t_2,x,w) \wedge \pi(z_1) = y = \pi(z_2))$ and the $q_{\pi(a)}$, where $z = (z_1,t_1,z_2,t_2)$, we obtain, for any $\pi(a), \pi(b)$ realizations of $\pi(q)$, that $\models \theta_{\pi(a)}(\overline{a}_1,\overline{b}_1,\overline{a}_2,\overline{b}_2,\pi(a))$ if and only if $\models \theta_{\pi(b)}(\overline{a}_1,\overline{b}_1,\overline{a}_2,\overline{b}_2,\pi(a))$. Thus, we can fix $\pi(b)$, and use the formula $\theta_{\pi(b)}$ to obtain for any $\pi(a) \models \pi(q)$, for any $\overline{a}_1, \overline{a}_2 \models r_{\pi(a)}$ and any $\overline{b}_1, \overline{b}_2$, that $(\overline{a}_1,\overline{b}_1)E(\overline{a}_2,\overline{b}_2)$ if and only if $\theta_{\pi(b)}(\overline{a}_1,\overline{b}_1,\overline{a}_2,\overline{b}_2,\pi(a))$. So $\theta_{\pi(b)}$ is the formula defining E.

$\square$

Hence we obtain an $\emptyset$-type-definable set X/E. But we had, by Claim 3.2.3, a map from X to $\mathrm{Mor}(\mathcal{G})$. And $(\overline{a}_1,\overline{b}_1)E(\overline{a}_2,\overline{b}_2)$ if and only if they have the same image under this map. Therefore we have obtained a bijection from X/E to $\mathrm{Mor}(\mathcal{G})$. Notice that this also yields $\emptyset$-definability of domain and codomain: since the maps are represented by elements in the fibers, we can just take images under π of any of their representative.

We can, using this coding for morphisms of the groupoid, prove that the groupoid action is relatively $\emptyset$-definable. If $\sigma \in \mathrm{Mor}(\pi(a),\pi(b))$, we can pick any representative $(\overline{a},\sigma(\overline{a}))$. Then $\sigma(a)$ is the unique tuple satisfying $\psi(\overline{a},\sigma(\overline{a}),a,z)$. Since this does not depend on the representative we pick, we obtain that $\sigma(a) \in \mathrm{dcl}(\sigma,a)$ (and the formula witnessing it is uniform in σ and a). This yields that the groupoid action is relatively $\emptyset$-definable.

To finish the proof, we need to construct the composition in an $\emptyset$-definable way.

Claim 3.2.6. *The composition of* $\mathrm{Mor}(\mathcal{G})$ *is definable.*

Proof. Let $\sigma, \tau, \mu \in \mathrm{Mor}(\mathcal{G})$. Let $\overline{a}, \overline{b}, \overline{c} \models r$, with $\pi(\overline{a}) = \mathrm{dom}(\sigma), \pi(\overline{b}) = \mathrm{dom}(\tau)$ and $\pi(\overline{c}) = \mathrm{dom}(\sigma)$. We will show that the equality $\tau \circ \sigma = \mu$ holds if and only if $\mathrm{dom}(\sigma) = \mathrm{dom}(\mu), \mathrm{cod}(\tau) = \mathrm{cod}(\mu), \mathrm{cod}(\sigma) = \mathrm{dom}(\tau)$ and for any $a \models q_{\pi(\overline{a})}|_{\overline{a},\overline{b},\overline{c},\sigma,\tau,\mu}$, we have:

$\forall z \psi(\overline{c}, \mu(\overline{c}), a, z) \leftrightarrow \exists u(\psi(\overline{a}, \sigma(\overline{a}), a, u) \wedge (\psi(\overline{b}, \tau(\overline{b}), u, z)))$

The left to right direction is again immediate. For the right to left direction, we can proceed as in Claim 3.2.5, and assume that the right-hand side holds. Pick $\overline{a}_2 \models r_{\pi(\overline{a})}|_{\sigma,\tau,\mu,\overline{a},\overline{b},\overline{c}}$, then, as was done in Claim 3.2.5, we obtain $\mu(\overline{a}_2) = \tau \circ \sigma(\overline{a}_2)$. But any $a' \models q_{\pi(\overline{a})}$ is equal to $f(\overline{a}_2, c)$ for some c tuple of realizations of $\mathcal{P}$. So we get $\mu(a') = f(\mu(\overline{a}_2), c) = f(\tau \circ \sigma(\overline{a}_2), c) = \tau \circ \sigma(a')$. So $\mu = \tau \circ \sigma$.

Note that since the type $q_{\pi(\overline{a})}$ is stationary and definable, the right-hand side condition is equivalent to a formula over $\mathrm{dom}(\sigma) = \pi(\overline{a})$. Moreover, the truth of this formula does not depend on the representants of σ, τ and μ that we pick. Therefore it only depends on σ, τ and μ.

Hence, if we fix $\pi(a)$, we obtain a formula $\theta_{\pi(a)}(x, y, z)$ over $\pi(a)$ such that for all $\sigma, \tau, \mu \in \mathrm{Mor}(\mathcal{G})$ with $\mathrm{dom}(\sigma) = \mathrm{dom}(\mu) = \pi(a)$, we have $\theta_{\pi(a)}(\sigma, \tau, \mu)$ if and only if $\mu = \tau \circ \sigma$. Again, this formula is a defining scheme for $q_{\pi(a)}$.

We can apply the proof of Claim 3.2.5 to this situation, to get a formula θ over $\emptyset$ such that for all $\sigma, \tau, \mu \in \mathrm{Mor}(\mathcal{G})$, $\mu = \tau \circ \sigma$ if and only if $\theta(\sigma, \tau, \mu)$. So the composition in $\mathcal{G}$ is relatively $\emptyset$-definable.

□

This finishes the proof: we have obtained a type-definable groupoid, and we already saw that its natural action on $q(\mathbb{M})$ is relatively $\emptyset$-definable. $\square$

Before we discuss how properties of the type and of this groupoid relate to each other, let us explain how this construction fits into the literature.

Let us start by noting that our groupoid is indeed a generalization of the binding group: if q is $\mathcal{P}$-internal, we can simply pick a constant $e \in \mathrm{dcl}(\emptyset)$ (which exists since $T = T^{\mathrm{eq}}$ and $\mathrm{acl}(\emptyset) = \emptyset$) and let $\pi(a) = e$ for any $a \models q$. Our theorem yields a one-object groupoid, that is, a group, which is the binding group of q over $\mathcal{P}$.

Groupoids related to internality have appeared before, under the name of binding groupoids. We will now point out the difference between the groupoid just constructed and binding groupoids.

Let us recall what these groupoids are, in one of the most tame context, which is definable sets in totally transcendental theories (see [38] for a proof):

Theorem 3.2.7 (Hrushovski)**.** *Let T be totally transcendental, let $\mathbb{M}$ be a monster model of T, and let $\phi(\mathbb{M})$ and $\psi(\mathbb{M})$ be two $\emptyset$-definable sets. Assume that $\phi(\mathbb{M})$ is ψ internal and non-empty. Then there is an $\emptyset$-definable connected groupoid $\mathcal{G}$ with a distinguished object a, and a full $\emptyset$-definable subgroupoid in $\psi(\mathbb{M})^{\mathrm{eq}}$. The isotropy group* $\mathrm{Mor}(a, a)$ *acts definably on $\phi(\mathbb{M})$, and this action is isomorphic to the action of* $\mathrm{Aut}(\phi(\mathbb{M})/\psi(\mathbb{M}))$ *on $\phi(\mathbb{M})$.*

Hence, this theorem allows us to view the binding group $\mathrm{Aut}(\phi(\mathbb{M})/\psi(\mathbb{M}))$ as an isotropy group, part of an $\emptyset$-definable groupoid.

The main point of the proof is that since $\phi(\mathbb{M})$ is $\psi(\mathbb{M})$-internal, there is a b-definable set O_b in $\psi(\mathbb{M})^{\mathrm{eq}}$ and a c-definable bijection $f_{c,b} : \phi(\mathbb{M}) \to O_b$. Ignoring issues of definability, the idea is now to allow these parameters to vary, the b yielding the objects of our groupoid and the c yielding, via the $f_{c,b}$, the morphisms.

Thus, this groupoid arises from the non-canonicity of the parameters used to witness internality of $\phi(\mathbb{M})$. The main point of constructing this groupoid from the binding group is to capture, canonically and intrinsically to $\phi(\mathbb{M})$, the data of an internal formula.

On the other hand, the groupoid we just constructed arises because if (q, π) is relatively $\mathcal{P}$-internal, the automorphism group $\mathrm{Aut}(q/\mathcal{P})$ and its action on q need not be (type-)definable. However, its action restricted to maps between two fibers is, and uniformly so. Thus, the point of constructing our groupoid is to allow binding groups to vary in families, and to capture the canonical algebraic object arising from these families. However, these groupoids are not *intrinsic* to $\mathcal{P}$, as they are not, in general, internal to $\mathcal{P}$.

The most recent developments regarding binding groupoids are the two papers [13] and [8]. In the first paper, Hrushovski works under the following set up: a sort $\mathbb{U}$, stably embedded in another sort $\mathbb{U}'$. The sort $\mathbb{U}'$ is assumed to be internal to $\mathbb{U}$, it is said that $\mathbb{U}'$ is an internal cover of $\mathbb{U}$. He proves (under some mild technical assumptions) that the group $\mathrm{Aut}(\mathbb{U}'/\mathbb{U})$ is part of an $\emptyset$-definable groupoid, and that there is a correspondence between certain definable groupoids in $\mathbb{U}$ and internal covers of $\mathbb{U}$. Haykazyan and Moosa expand on Hrushovski's results, and also prove that the previously mentioned correspondence is an equivalence of categories.

Hence internal covers of $\mathbb{U}$ are witnessed by objects living in $\mathbb{U}$, and in particular, their binding group is $\mathbb{U}$-internal. Thinking about analysability, a natural question is then to ask what canonical object, intrisic to $\mathbb{U}$, captures the data of a varying family of internal covers. To properly answer this question, the authors introduce 1-analysable covers over some set A. They define a 1-analysable cover as a cover $\mathbb{U}'$ of $\mathbb{U}$ with a single new sort S, equipped with an $\emptyset$-definable surjective map $\pi : S \to A$ such that each fiber $\pi^{-1}(a)$ is internal to $\mathbb{U}$.

In [8], Haykazyan and Moosa prove that under a specific assumption of independence of the fibers, Hrushovski's groupoid construction is uniformly definable across fibers, giving rise to a definable groupoid from a 1-analysable cover. This is the first step in finding out what happens when Hrushovski's groupoids vary in families. In the case of non-independent fibers, Hurshovski suggests in [13] that the relevant algebraic object is a definable simplicial groupoid.

Our work here explores a similar topic. Indeed, notice the similarity between our notion of relatively internal pair (q, π) and 1-analysable covers, which correspond to the case where $\pi(q)$ is $\mathcal{P}$-internal too. We will, in section 3.3, obtain a type-definable Delta groupoid from any relatively internal pair, and even a simplicial groupoid if one is willing to drop uniform definability of the isotropy groups. However, our groupoid once again arises for different reasons, and encodes varying binding *groups*, but binding *groupoids*, encoding non-canonicity of parameters, do not appear.

Thus, our work is in a way orthogonal to Haykazyan, Hrushovski and Moosa's: we ignore issues of finding intrinsic binding groups, and instead focus on varying families of binding groups. The fact that in both cases, the relevant algebraic object is a groupoid could indicate that there is a unified approach here.

However, there is an obstruction to this: one key property of Hrushovski's binding groupoids is that, keeping the same notation, they are internal to the sort $\mathbb{U}$. Unfortunately, the groupoid we obtain is, in some cases, not internal to the family $\mathcal{P}$, as the following result, obtained in collaboration with Omar Léon Sánchez, shows:

Proposition 3.2.8. *If $\mathcal{G}$ is internal to $\mathcal{P}$ and connected, then q is internal to $\mathcal{P}$.*

Proof. By internality assumption, there is a set of parameters B such that $\mathrm{Mor}(\mathcal{G}) \subset \mathrm{dcl}(\mathcal{P}, B)$.

Let a and b be any realizations of q, and let $\overline{a}$ be a fundamental system of solutions for $\mathrm{tp}(a/\pi(a))$. Since $\mathcal{G}$ is connected, there is $\sigma \in \mathrm{Mor}(\pi(a), \pi(b))$. Moreover, the tuple $\overline{b} = \sigma(\overline{a})$ is a fundamental system of solutions for $\mathrm{tp}(b/\pi(b))$. Therefore, there is $\overline{d} \in \mathcal{P}$ such that $b = f(\overline{b}, \overline{d}) = f(\sigma(\overline{a}), \overline{d})$. But $\sigma \in \mathrm{dcl}(\mathcal{P}, B)$, therefore $b \in \mathrm{dcl}(\mathcal{P}, B, \overline{a})$. $\square$

Hence the question of which object living in $\mathcal{P}$, if any, captures the data of a relatively internal pair, is still open.

3.3 Various properties of the groupoid, Delta groupoids

This section is dedicated to exploring the relationship between a relatively internal pair (q, π) and its groupoid $\mathcal{G}(q, \pi, \mathcal{P})$. We will start by examining a very strong property of the groupoid, called retractability.

3.3.1 Retractability

Retractability was introduced in [7] by Goodrick and Kolesnikov, let us recall their definition:

Definition 3.3.1. An $\emptyset$-type-definable groupoid $\mathcal{G}$ is retractable if it is connected and there exist an $\emptyset$-definable partial function $g(x, y) = g_{x,y}$ such that for all a, b objects of $\mathcal{G}$, we have $g_{a,b} \in \mathrm{Mor}(a, b)$. Moreover, we require the compatibility condition that $g(b, c) \circ g(a, b) = g(a, c)$ for all objects a, b, c (note that this implies $g_{a,a} = \mathrm{id}_a$ and $g_{a,b}^{-1} = g_{b,a}$ for all a, b).

In their paper, they use retractability to study groupoids arising from internality, and uncover a link between retractability and 3-amalgamation. As it turns out, retractability will have very strong consequences for relatively internal types.

Before delving into these, let us give an equivalent definition, which will be useful to us:

Remark 3.3.2. An equivalent definition of retractability is given by: there exist an $\emptyset$-type-definable group G, and a full, faithful $\emptyset$-definable functor $F : \mathcal{G} \to G$.

This was proved in [7], we include their proof here for completeness:

Proof. If we have such a functor $F : \mathcal{G} \to G$, we can take $g_{a,b} = F^{-1}(\{\mathrm{id}_G\}) \cap \mathrm{Mor}(a,b)$, which is a singleton because F is full and faithful. The compatiblity condition is easily checked, and this is definable uniformly in (a,b).

If $\mathcal{G}$ is retractable, then we can construct a relation E on $\mathrm{Mor}(\mathcal{G})$ as follows: if $\sigma \in \mathrm{Mor}(a,b)$ and $\tau \in \mathrm{Mor}(c,d)$, then $E(\sigma,\tau)$ if and only if $\tau = g_{b,d} \circ \sigma \circ g_{c,a}$. By the compatibility condition, this is an equivalence relation, and it is $\emptyset$-definable. Now consider $G = \mathrm{Mor}(\mathcal{G})/E$, and $F : \mathcal{G} \to G$ the quotient map. The groupoid law of $\mathcal{G}$ goes down to a group law on G. Indeed, if we want to compose $\sigma \in \mathrm{Mor}(a,b)$ and $\tau \in \mathrm{Mor}(c,d)$ in G, notice that $E(\tau, g_{d,a} \circ \tau)$, so we can define $F(\sigma) \circ F(\tau) = F(\sigma \circ g_{d,a} \circ \tau)$. Again by the compatibility condition, this is well defined. Finally, it is easy to derive the group axioms from the groupoid axioms of $\mathcal{G}$. $\square$

For the rest of this subsection, we again assume $\mathrm{acl}(\emptyset) = \emptyset$, fix a type $q \in S(\emptyset)$, an $\emptyset$-definable map π, and a family of types $\mathcal{P}$ over $\emptyset$, such that (q,π) is relatively $\mathcal{P}$-internal. We consider the $\emptyset$-type-definable groupoid $\mathcal{G} = \mathcal{G}(q, \pi/\mathcal{P})$ constructed in the previous section.

The definition of retractability asks for the groupoid to be connected. Thus, before delving into the consequences of retractability, it is natural to ask what does connectedness of $\mathcal{G}(q, \pi/\mathcal{P})$ entails. Following a suggestion of Rahim Moosa, we do so here. Recall that for a type $p \in S(\emptyset)$, we can always consider the group $\mathrm{Aut}(p/\mathcal{P})$ of automorphisms of $p(\mathbb{M})$ fixing $\mathcal{P}$ pointwise and induced by automorphisms of $\mathbb{M}$, even if p is not $\mathcal{P}$-internal and $\mathrm{Aut}(p/\mathcal{P})$ is not type-definable. The following proposition is well-known, but we include a proof here for completeness.

Proposition 3.3.3. *For any type $p \in S(\emptyset)$, the group $\mathrm{Aut}(p/\mathcal{P})$ acts transitively on $p(\mathbb{M})$ if and only if p is weakly orthogonal to $\mathcal{P}$.*

Proof. If p is weakly orthogonal to $\mathcal{P}$, then for any $a, b \models p$, both a and b are independent, over $\emptyset$, of any small subset of $\mathcal{P}$. In particular, we have that $a \mathop{\smile\!\!\!\!\!\!\mid}\, (\mathrm{dcl}(a,b)) \cap \mathcal{P}$

and $b \mathop{\smile\!\!\!\!\mid}\ (\mathrm{dcl}(a,b)) \cap \mathcal{P}$, hence by stationarity $\mathrm{tp}(a/(\mathrm{dcl}(a,b)) \cap \mathcal{P}) = \mathrm{tp}(b/(\mathrm{dcl}(a,b)) \cap \mathcal{P})$. Thus, by Fact 2.2.15, we obtain $\mathrm{tp}(a/\mathcal{P}) = \mathrm{tp}(b/\mathcal{P})$. By Fact 2.2.14, this means that there is $\sigma \in \mathrm{Aut}(p/\mathcal{P})$ such that $\sigma(a) = b$. Thus $\mathrm{Aut}(p/\mathcal{P})$ acts transitively on $p(\mathbb{M})$.

If p is not weakly orthogonal to $\mathcal{P}$, there exist $a \models p$ and a tuple c of realizations of $\mathcal{P}$ such that a forks with c over $\emptyset$. Taking $b \models p|_c$, we see that $\mathrm{tp}(a/\mathcal{P}) \neq \mathrm{tp}(b/\mathcal{P})$, thus a and b have different $\mathrm{Aut}(p/\mathcal{P})$ orbits, and the action cannot be transitive.

□

Using this proposition, we obtain:

Corollary 3.3.4. *The groupoid $\mathcal{G}(q, \pi/\mathcal{P})$ is connected if and only if $\pi(q)$ is weakly orthogonal to $\mathcal{P}$.*

Proof. By the previous proposition, it is enough to prove that $\mathcal{G} = \mathcal{G}(q, \pi/\mathcal{P})$ is connected if and only if $\mathrm{Aut}(\pi(q)/\mathcal{P})$ acts transitively on $\pi(q)(\mathbb{M})$.

Let $\pi(a), \pi(b)$ be any two realizations of $\pi(q)$. If $\mathcal{G}$ is connected, then there exist a morphism $\sigma \in \mathrm{Mor}(\pi(a), \pi(b))$. It is induced by an automorphism σ of $\mathbb{M}$ fixing $\mathcal{P}$ pointwise. In particular it restricts to $\sigma \in \mathrm{Aut}(\pi(q)/\mathcal{P})$ such that $\sigma(\pi(a)) = \pi(b)$.

Conversely, if $\mathrm{Aut}(\pi(q)/\mathcal{P})$ acts transitively on $\pi(q)(\mathbb{M})$, then for any $\pi(a), \pi(b)$ realizations of $\pi(q)$, there is an automorphism σ of $\mathbb{M}$, fixing $\mathcal{P}$ pointwise, such that $\sigma(\pi(a)) = \pi(b)$. This automorphism σ restricts to an element of $\mathrm{Mor}(\pi(a), \pi(b))$.

□

In particular, retractability of $\mathcal{G}(q, \pi/\mathcal{P})$ implies that $\pi(q)$ is weakly orthogonal to $\mathcal{P}$. As we will see, it actually imposes much stronger restrictions on (q, π). Our first result links retractability to products of types:

Theorem 3.3.5. *The groupoid $\mathcal{G}$ is retractable if and only if there exists a type $p \in S(\emptyset)$, internal to $\mathcal{P}$, with $\pi(q)$ weakly orthogonal to the family $\mathcal{P} \cup \{p\}$, and such that $q = p \otimes \pi(q)$ (up to an $\emptyset$-definable bijection).*

Proof. If $\mathcal{G}$ is retractable, consider the $\emptyset$-definable relation $xEy \Leftrightarrow g_{\pi(x),\pi(y)}(x) = y$. The compatibility condition of retractability implies that this is an equivalence relation. Let ρ be the quotient map, then $\rho(q)$ is a complete type over the empty set, and it will be the type p of the theorem.

There is an $\emptyset$-definable function $s : q(\mathbb{M}) \to \rho(q)(\mathbb{M}) \times \pi(q)(\mathbb{M})$ sending x to $(\rho(x), \pi(x))$. Since q is a complete type, $s(q(\mathbb{M}))$ is the set of realizations of a complete type, denoted $s(q)$. But the function s is bijective. Indeed, notice that each E-class has exactly one element in each fiber of π: each class has at least one element in a given fiber because $\mathcal{G}$ is connected, and no more than one because $g_{\pi(a),\pi(a)} = \mathrm{id}_{\pi(a)}$. Therefore we can send $(\rho(a), \pi(b))$ to the unique element both in the $\pi(b)$ fiber and in the E-class of a, to obtain an inverse of s. In particular $\pi(q)$ and $\rho(q)$ are weakly orthogonal, so $\rho(q)(\mathbb{M}) \times \pi(q)(\mathbb{M}) = \rho(q) \otimes \pi(q)(\mathbb{M})$. We denote $p = \rho(q)$.

Note that for any $\pi(a), \pi(b) \models \pi(q)$, the morphism $g_{\pi(a),\pi(b)}$ extends to an automorphism of $\mathbb{M}$, fixing $\mathcal{P}(\mathbb{M}) \cup p(\mathbb{M})$ pointwise, and sending $\pi(a)$ to $\pi(b)$. Thus, by Proposition 3.3.3, we obtain that $\pi(q)$ is weakly orthogonal to $\mathcal{P} \cup p$.

We now just need to prove that p is $\mathcal{P}$-internal. Each E-class has a unique representant in each π-fiber. Therefore, fixing $a \models q$, we have $p(\mathbb{M}) \subset \mathrm{dcl}(q_{\pi(a)}(\mathbb{M}))$. But by internality of the fibers, we get $q_{\pi(a)}(\mathbb{M}) \subset \mathrm{dcl}(\overline{a}, \mathcal{P})$, for some tuple $\overline{a}$. This yields $p(\mathbb{M}) \subset \mathrm{dcl}(\overline{a}, \mathcal{P})$.

Conversely, suppose that $q = p \otimes \pi(q)$. As $\pi(q)$ is weakly orthogonal to $\mathcal{P} \cup \{p\}$, for any $\pi(a), \pi(b) \models \pi(q)$, there is $\sigma \in \mathrm{Aut}(\mathbb{M})$, fixing $\mathcal{P}(\mathbb{M}) \cup p(\mathbb{M})$ pointwise, and such that $\sigma(\pi(a)) = \pi(b)$. Moreover, the restriction of such a σ to an element of $\mathrm{Mor}(\pi(a), \pi(b))$ is unique, denote it $g_{\pi(a),\pi(b)}$. The family $g_{\cdot,\cdot}$ witnesses retractability of $\mathcal{G}$.

□

Theorem 3.3.5 immediately yields the following internality criteria:

Corollary 3.3.6. *If $\mathcal{G}$ is retractable and $\pi(q)$ is $\mathcal{P}$-internal, then q is $\mathcal{P}$-internal.*

Note that this is far from a necessary and sufficient condition. We will give a counterexample later in this section.

Recall that retractability yields a definable, full and faithful functor $F : \mathcal{G} \to G$, for some $\emptyset$-type-definable group G. This functor will allow us to construct a groupoid morphism from $\mathrm{Aut}(q/\mathcal{P}) \to G$, even when the former group is not type-definable.

Proposition 3.3.7. *If $\mathcal{G}$ is retractable, there is a morphism $R : \mathrm{Aut}(q/\mathcal{P}) \to G$, which is surjective.*

Proof. We use the functor $F : \mathcal{G} \to G$. For $\sigma \in \mathrm{Aut}(q/\mathcal{P})$, note that the restriction of σ to $q_{\pi(a)}(\mathbb{M})$ is an element of $\mathrm{Mor}(\pi(a), \sigma(\pi(a)))$. We denote it by $\sigma|_{\pi(a)}$. We can then set $R(\sigma) = F(\sigma|_{\pi(a)})$. Let us show that R is a surjective morphism $R : \mathrm{Aut}(q/\mathcal{P}) \to G$.

First, we need to prove that R is well defined. To do so, we need to show that for any b, we have $\sigma|_{\pi(b)} = g_{\sigma(\pi(a)),\sigma(\pi(b))} \circ \sigma|_{\pi(a)} \circ g_{\pi(b),\pi(a)}$, by definition of F.

Pick any x with $\pi(x) = \sigma(\pi(a))$. As $g_{\_,\_}$ is an uniformly $\emptyset$-definable family of partial functions, we have $g_{\sigma(\pi(a)),\sigma(\pi(b))}(x) = y$ if and only if $g_{\pi(a),\pi(b)}(\sigma^{-1}(x)) = \sigma^{-1}(y)$, for any y. Applying σ to the second equality, we get, for all y, that $g_{\sigma(\pi(a)),\sigma(\pi(b))}(x) = y$ if and only if $\sigma(g_{\pi(a),\pi(b)}(\sigma^{-1}(x))) = y$, which yields that $\sigma|_{\pi(b)} \circ g_{\pi(a),\pi(b)} \circ \sigma|_{\pi(a)}^{-1} = g_{\sigma(\pi(a)),\sigma(\pi(b))}$, what we wanted.

Therefore we have a well defined map $R : \mathrm{Aut}(q/\mathcal{P}) \to G$. It is a morphism because:

$$\begin{aligned} R(\sigma \circ \tau) &= F((\sigma \circ \tau)|_{\pi(a)}) \\ &= F(\sigma|_{\tau(\pi(a))} \circ \tau|_{\pi(a)}) \\ &= F(\sigma|_{\tau(\pi(a))}) \circ F(\tau|_{\pi(a)})) \\ &= R(\sigma) \circ R(\tau) \end{aligned}$$

For surjectivity, it is enough to prove that for $\sigma \in \mathrm{Mor}(\pi(a), \pi(a))$, that there is $\tau \in \mathrm{Aut}(q/\mathcal{P})$ restricting to σ. This is true by definition of $\mathrm{Mor}(\pi(a), \pi(a))$.

□

Theorem 3.3.5 yielded the existence of a $\mathcal{P}$-internal type $p \in S(\emptyset)$, weakly internal to $\mathcal{P} \cup \{\pi(q)\}$, such that $q(\mathbb{M})$ is in $\emptyset$-definable bijection with $p \otimes \pi(q)(\mathbb{M})$. The group G is none other that the binding group of p:

Proposition 3.3.8. *The group G witnessing retractability is relatively $\emptyset$-definably isomorphic to $\mathrm{Aut}(p/\mathcal{P})$, the binding group of p over $\mathcal{P}$ (where p is the type of Theorem 3.3.5).*

Proof. Recall that $\mathrm{Mor}(\mathcal{G})$ is given by X/E, where X is an $\emptyset$-type-definable set, and E is an $\emptyset$-definable equivalence relation. Moreover, the type-definable set X is composed of pairs of realizations of r, the type introduced in the proof of Theorem 3.1.3. In the proof Theorem 3.3.5, we constructed an $\emptyset$-definable quotient map $\rho : q(\mathbb{M}) \to p(\mathbb{M})$. The type $p = \rho(q)$ is $\mathcal{P}$-internal, hence its binding group $\mathrm{Aut}(p/\mathcal{P})$ is similarly given by the type r' of a fundamental system of solutions, an $\emptyset$-type-definable set X' and an $\emptyset$-definable equivalence relation E'. We can assume that $r' = \rho(r)$.

For any $\overline{a} \models r$, this allows us to define a group morphism:

$$P_{\pi(\overline{a})} : \mathrm{Aut}(\mathrm{tp}(\overline{a}/\pi(\overline{a}))/\mathcal{P}) \to \mathrm{Aut}(p/\mathcal{P})$$
$$\sigma = (\overline{a}, \sigma(\overline{a}))/E \to (\rho(\overline{a}), \rho(\sigma(\overline{a})))/E'$$

and by construction of ρ, this is an isomorphism. It is relatively $\overline{a}$-definable.

We are also given, by the retractability assumption, a relatively $\emptyset$-definable full and faithful functor $F : \mathrm{Mor}(\mathcal{G}) \to G$. By restriction this yields, for any $\overline{a} \models r$, a relatively $\pi(\overline{a})$-definable group isomorphism $F_{\pi(\overline{a})} : \mathrm{Aut}(\mathrm{tp}(\overline{a}/\pi(\overline{a}))/\mathcal{P}) \to G$.

Hence, for any $\overline{a} \models r$, the groups G and $\mathrm{Aut}(p/\mathcal{P})$ are relatively $\overline{a}$-definably isomorphic via the composition $P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}$. To complete the proof, we need to show that this morphism is actually relatively $\emptyset$-definable. To do so, it is enough (via a compactness argument) to prove that the graph of $P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}$ is fixed by any automorphism of $\mathbb{M}$.

Claim 3.3.9. *For any $\overline{a}, \overline{b} \models r$ and $g \in G$, we have $P_{\pi(\overline{b})} \circ F^{-1}_{\pi(\overline{b})}(g) = P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}(g)$.*

Proof. By the proof of Proposition 3.3.7, if $\overline{a}, \overline{b}$ are realizations of r and $g \in G$, then there is $\sigma \in \mathrm{Aut}(q/\mathcal{P})$ such that $F^{-1}_{\pi(\overline{a})}(g)$ is the restriction of σ to $\mathrm{Aut}(\mathrm{tp}(\overline{a}/\pi(\overline{a}))/\mathcal{P})$ and $F^{-1}_{\pi(\overline{b})}(g)$ is the restriction of σ to $\mathrm{Aut}(\mathrm{tp}(\overline{b}/\pi(\overline{b}))/\mathcal{P})$. So $F^{-1}_{\pi(\overline{a})}(g) = (\overline{a}, \sigma(\overline{a}))/E$ and $F^{-1}_{\pi(\overline{b})}(g) = (g_{\pi(\overline{a}),\pi(\overline{b})}(\overline{a}), \sigma(g_{\pi(\overline{a}),\pi(\overline{b})}(\overline{a})))/E$, as $g_{\pi(\overline{a}),\pi(\overline{b})}(\overline{a}) \models r$. We then obtain:

$$\begin{aligned}
P_{\pi(\overline{b})} \circ F^{-1}_{\pi(\overline{b})}(g) &= P_{\pi(\overline{b})}((g_{\pi(\overline{a}),\pi(\overline{b})}(\overline{a}), \sigma(g_{\pi(\overline{a}),\pi(\overline{b})}(\overline{a})))/E) \\
&= (\rho(g_{\pi(\overline{a}),\pi(\overline{b})}(\overline{a})), \rho(\sigma(g_{\pi(\overline{a}),\pi(\overline{b})}(\overline{a}))))/E' \\
&= (\rho(g_{\pi(\overline{a}),\pi(\overline{b})}(\overline{a})), \sigma(\rho(g_{\pi(\overline{a}),\pi(\overline{b})}(\overline{a}))))/E' \\
&= (\rho(\overline{a}), \sigma(\rho(\overline{a})))/E' \text{ by definition of } \rho \\
&= P_{\pi(\overline{a})}(\overline{a}, \sigma(\overline{a})/E) \\
&= P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}(g)
\end{aligned}$$

□

Now let $g \in G$, let $(g, P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}(g))$ be in the graph of $P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}$, and let μ be an automorphism of $\mathbb{M}$. We want to show that $\mu(g, P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}(g))$ is also in the graph of $P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}$.

Claim 3.3.10. *We have $\mu(F_{\pi(\overline{a})}) = F_{\pi(\mu(\overline{a}))}$.*

Proof. This is because the maps $F^{-1}_{\pi(\overline{a})}$ are uniformly $\pi(\overline{a})$-definable.

□

Claim 3.3.11. *For* $\sigma \in \mathrm{Aut}(\mathrm{tp}(\overline{a}/\pi(\overline{a}))/\mathcal{P})$*, we have* $\mu(P_{\pi(\overline{a})}(\sigma)) = P_{\pi(\mu(\overline{a}))}(\mu(\sigma))$*.*

Proof. The set $\mathrm{Mor}(\mathcal{G})$ is $\emptyset$-type-definable, hence for any $\sigma \in \mathrm{Aut}(\mathrm{tp}(\overline{a}/\pi(\overline{a}))/\mathcal{P})$ we have $\mu(\sigma) = \tau \in \mathrm{Mor}(\mathcal{G})$. In particular, we obtain $\mu(\sigma(\overline{a})) = \tau(\mu(\overline{a}))$, which yields:

$$\begin{aligned}
\mu(P_{\pi(\overline{a})}(\sigma)) &= \mu(P_{\pi(\overline{a})}((\overline{a}, \sigma(\overline{a}))/E)) \\
&= (\rho(\mu(\overline{a})), \rho(\mu(\sigma(\overline{a}))))/E' \\
&= (\rho(\mu(\overline{a})), \rho(\tau(\mu(\overline{a}))))/E' \\
&= P_{\pi(\mu(\overline{a}))}((\mu(\overline{a})), \tau(\mu(\overline{a}))/E) \\
&= P_{\pi(\mu(\overline{a}))}(\tau) \\
&= P_{\pi(\mu(\overline{a}))}(\mu(\sigma))
\end{aligned}$$

□

Putting everything together, we obtain:

$$\begin{aligned}
\mu(P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}(g)) &= \mu(P_{\pi(\overline{a})}) \circ \mu(F_{\pi(\overline{a})})^{-1}(\mu(g)) \\
&= P_{\pi(\mu(\overline{a}))} \circ F^{-1}_{\pi(\mu(\overline{a}))}(\mu(g)) \text{ by Claims 3.3.10 and 3.3.11} \\
&= P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}(\mu(g)) \text{ by Claim 3.3.9}
\end{aligned}$$

so $(\mu(g), P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}(\mu(g)))$ belongs to the graph of $P_{\pi(\overline{a})} \circ F^{-1}_{\pi(\overline{a})}$, what we needed to prove.

□

Remark that by examining the proof of Theorem 3.3.5, we see that G is also isomorphic to $\mathrm{Aut}(q_{\pi(a)}/\mathcal{P})$, for any $\pi(a) \models \pi(q)$, i.e. the binding group of a fiber.

Equipped with all our previous observations, we are now ready to prove the strongest structural result of this section. Recall that a $\mathcal{P}$-internal type q is said to be fundamental if it has a fundamental system of solutions consisting of one $a \models q$, that is, we have $q(\mathbb{M}) \subset \mathrm{dcl}(a, \mathcal{P})$.

Theorem 3.3.12. *If $\mathcal{G}$ is retractable and $\pi(q)$ is $\mathcal{P}$-internal and fundamental, then q is $\mathcal{P}$-internal and $\mathrm{Aut}(q/\mathcal{P})$ is $\emptyset$-definably isomorphic to $G \times \mathrm{Aut}(\pi(q)/\mathcal{P})$.*

Proof. We know from Corollary 3.3.6 that q is $\mathcal{P}$-internal. Let $\overline{a}$ be a fundamental system of solutions for q.

Recall that there are two $\emptyset$-definable quotient maps $\pi : q(\mathbb{M}) \to \pi(q)(\mathbb{M})$ and $\rho : q(\mathbb{M}) \to p(\mathbb{M}) = \rho(q)(\mathbb{M})$. The tuples $\pi(\overline{a})$ and $\rho(\overline{a})$ are fundamental systems of solutions for $\pi(q)$ and $\rho(q)$. As was done in Proposition 3.3.8, we can use this to construct two $\overline{a}$-definable surjective group morphisms $\overline{\pi} : \mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\pi(q)/\mathcal{P})$ and $\overline{\rho} : \mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\rho(q)/\mathcal{P})$. Using techniques similar to the ones in Proposition 3.3.8, we can prove that these two morphisms are $\emptyset$-definable.

Hence we have produced two $\emptyset$-definable group morphisms $\overline{\pi} : \mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\pi(q)/\mathcal{P})$ and $\overline{\rho} : \mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\rho(q)/\mathcal{P})$, both surjective. To obtain the desired isomorphism, it would be enough to prove that $\ker(\overline{\pi}) \cap \ker(\overline{\rho}) = \mathrm{id}$ and that any element of $\mathrm{Aut}(q/\mathcal{P})$ can be written as the product of an element of $\ker(\overline{\rho})$ and an element of $\ker(\overline{\pi})$.

Suppose that $\sigma \in \ker(\overline{\pi}) \cap \ker(\overline{\rho})$, and let $a \models q$. Then σ fixes $\pi^{-1}\{\pi(a)\}$ setwise. But $\sigma \in \ker(\overline{\rho})$, hence must fix $\pi^{-1}\{\pi(a)\}$ pointwise. Since this is true for any $a \models q$, we conclude that $\sigma = \mathrm{id}$.

Let σ be any morphism in $\mathrm{Aut}(q/\mathcal{P})$ and $a \models q$. Consider $g_{\pi(a),\pi(\sigma(a))} \in \mathcal{G}$. It extends to an automorphism $\tau \in \mathrm{Aut}(q/\mathcal{P})$ by Fact 2.2.14, which has to belong to $\ker(\overline{\rho})$. We can write $\sigma = \tau \circ \tau^{-1} \circ \sigma$, so we only need to prove that $\tau^{-1} \circ \sigma \in \ker(\overline{\pi})$. But $\pi(\tau^{-1} \circ \sigma(a)) = \pi(a)$ and $\pi(q)$ is fundamental, so this implies $\overline{\pi}(\tau^{-1} \circ \sigma) = \mathrm{id}$.

□

The assumption that $\pi(q)$ is a fundamental type may or may not be necessary to get this result.

Let us put Theorem 3.3.12 to use by showing that internality of q does not imply retractability of $\mathcal{G}$:

Example 3.3.13. Consider the two sorted structure $\mathbb{M} = (G, X, \mathcal{L}_G, *)$ with one sort being a connected stable group G in the language $\mathcal{L}_G$, and the other sort being a principal homogeneous space X for G, with group action $*$. We will work in $\mathbb{M}^{eq}$, and the language is $\mathcal{L}_G \cup \{*\}$.

One can quickly prove that the sort X has only one 1-type q over $\emptyset$, and that this type is stationary and internal to G, with binding group isomorphic to G.

Assume that there is an $\emptyset$-definable normal subgroup H of G, such that the short exact sequence:

$1 \to H \to G \to G/H \to 1$

does not definably split.

The group action of G on X defines an equivalence relation E, where the class of an element $a \in X$ is its orbit $H*a$. Hence, we can define a map $\pi : X \to X/E$, sending $a \in X$ to $H * a$. This is $\emptyset$-definable, we have $\mathrm{Aut}(\pi(q)/G) \cong G/H$ and for any a, that $\mathrm{Aut}(\mathrm{tp}(a/\pi(a)) = H$. The pair (q, π) is relatively G-internal, yielding a groupoid $\mathcal{G}$. Since G/H acts transitively on $\pi(q)$, this groupoid is connected. Moreover, we have a definable short exact sequence:

$1 \to H \to G \to G/H \to 1$

which, by assumption, is not definably split. However, if $\mathcal{G}$ was retractable, Theorem 3.3.12 implies that this sequence would be definably split. Hence $\mathcal{G}$ is not retractable, even though q is G-internal.

Hence retractability is a very strong property of the groupoid: if $\pi(q)$ is internal, it implies much more than internality of q. In the next section, we will state a necessary and sufficient condition, in terms of groupoids, for q to be internal. However, the proof will have to wait for the next chapter, as we will then have better tools.

3.3.2 The Simplicial Approach

In [13], Hrushovski suggests capturing the data of a relatively internal cover (defined just like relatively internal types: a projection with internal fibers) via a definable simplicial groupoid. Inspired by his suggestion, we will construct, in this section, a type-definable Delta groupoid from any relatively internal pair. We will also show that while a simplicial groupoid is associated to any relatively internal pair, it fails to be type-definable. Let us note once again that our groupoids do not answer Hrushovski's question, as they do not arise from non-canonicity of parameters. We refer the reader to the discussion at the end of Section 3.2 for more details.

Let us start by defining Delta groupoids, which have less structure than simplicial groupoids.

Definition 3.3.14. The category $\hat{\Delta}$ is the category with finite ordinals as objects, and *strictly* order preserving functions as morphisms.

We can now give the very concise categorical definition of a Delta groupoid:

Definition 3.3.15. Given a category $\mathcal{C}$, a Delta object in $\mathcal{C}$ is a functor $F : \hat{\Delta}^{op} \to \mathcal{C}$.

A Delta groupoid is a Delta object in the category of groupoids.

For our purposes, the following equivalent definition will be much easier to handle:

Proposition 3.3.16. *A Delta groupoid is equivalent to the following data :*

1. *For every integer* $n \in \mathbb{N}\backslash\{0\}$*, a groupoid* $\mathcal{G}_n$
2. *For every integer* $n \in \mathbb{N}\backslash\{0, 1\}$*, and every* $i \in \{0, \cdots, n\}$*, a groupoid morphism (that is, a functor)* $\partial_i^n : \mathcal{G}_n \to \mathcal{G}_{n-1}$*, called a face map*

subject to the following condition :

$\partial_i^n \circ \partial_j^{n+1} = \partial_{j-1}^n \circ \partial_i^{n+1}$ *for all* $i < j \leq n$ *and* $n \geq 1$.

For a gentle introduction to Delta sets and simplicial sets, we refer the reader to [6].

Of course, we need to specify what we mean by a (type-)definable Delta groupoid:

Definition 3.3.17. Let A be some set of parameters. A Delta groupoid $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is said to be A-(type-)definable if each groupoid $\mathcal{G}_n$ and face map ∂_i^n is A-(type-)definable.

Remark that we do not ask for uniform definability of the groupoids. And in fact, in our case, the groupoids will not be uniformly definable. Let us now turn to the construction of a type-definable Delta groupoid.

Recall that we are working with a family of partial types $\mathcal{P}$ over the empty set, a type $q \in S(\emptyset)$, and an $\emptyset$-definable function π such that (q, π) is relatively $\mathcal{P}$-internal.

We obtained an $\emptyset$-type-definable groupoid $\mathcal{G} = \mathcal{G}(q, \pi/\mathcal{P})$ from this configuration. For each n, we can also consider the product of q with itself n times, denoted $q^{\otimes n}$. It is the type of n independent realizations of q. For π we get another definable map $\pi^{\otimes n} : q^{\otimes n} \to \pi(q)^{\otimes n}$, by applying π to each of the coordinates. The fibers of $\pi^{\otimes n}$ are of course $\mathcal{P}$-internal. Finally, for each $\overline{a} \models q^{\otimes n}$, an easy application of forking calculus yields that $\text{tp}(\overline{a}/\pi(\overline{a}))$ is stationary. Thus:

Observation 3.3.18. *The pair* $(q^{\otimes n}, \pi^{\otimes n})$ *is relatively* $\mathcal{P}$*-internal.*

Hence Theorem 3.1.3 yields a sequence of groupoids $\mathcal{G}_n = \mathcal{G}(q^{\otimes n}, \pi^{\otimes n}/\mathcal{P})$, with $\mathcal{G}_1 = \mathcal{G}(q, \pi/\mathcal{P})$. These will be part of our type-definable Delta groupoid, and we now only need to construct the face maps.

Intuitively, the groupoid $\mathcal{G}_n$ represent the action of $\text{Aut}(\mathbb{M}/\mathcal{P})$ on the product of n independent fibers of π. Therefore one's first instinct is to define the face maps as restriction maps. Luckily, this is correct, the only difficulty being to make sure these are definable.

To do so, we will need to dive back into the construction of $\mathcal{G}_1$. Recall that we started with r, the type of a fundamental system of solutions for the $\mathcal{P}$-internal type q. We then considered the $\emptyset$-type-definable set $X = \{(\overline{a}, \overline{b}), \overline{a} \models r, \overline{b} \models r, \text{tp}(\overline{a}/\mathcal{P}) = \text{tp}(\overline{b}/\mathcal{P})\}$, and proved that there was a relatively $\emptyset$-definable equivalence relation E on X such that $\text{Mor}(\mathcal{G})$ is in bijection with X/E.

Crucially, for each n, the type $r^{\otimes n}$ is a fundamental type for $q^{\otimes n}$. Hence, the construction of Theorem 3.1.3 goes through using $r^{\otimes n}$. More precisely, if we define $X_n = \{(\overline{a}, \overline{b}), \overline{a} \models r^{\otimes}, \overline{b} \models r^{\otimes n}, \text{tp}(\overline{a}/\mathcal{P}) = \text{tp}(\overline{b}/\mathcal{P})\}$, there is a relatively $\emptyset$-definable equivalence relation E_n on X_n such that $\text{Mor}(\mathcal{G}_n)$ is in bijection with X_n/E_n.

This yields $\emptyset$-definable functors between the $\mathcal{G}_n$. To see this, let us introduce some notation: if $\overline{a} = (a_1, \cdots, a_n)$ is a tuple, then for any $1 \leq i \leq n$, we denote $\overline{a}^{\wedge i} = (a_1, \cdots, \widehat{a_i}, \cdots a_n)$ where the hat means the corresponding coordinate has been removed. Now, if $n > 1$, an element σ of $\text{Mor}(\mathcal{G}_n)$ corresponds to the E_n-class of $(\overline{a}, \overline{b}) = ((a_1, \cdots, a_n), (b_1, \cdots, b_n))$, where $\overline{a}$ and $\overline{b}$ are realizations of $r^{\otimes n}$. For any $1 \leq i \leq n$, we can then send $(\overline{a}, \overline{b})/E_n$ to $(\overline{a}^{\wedge i}, \overline{b}^{\wedge i})/E_{n-1}$. This is well defined, as $(\overline{a}^{\wedge i}, \overline{b}^{\wedge i}) \in X_{n-1}$, and $\emptyset$-definable.

For each $n > 1$ and each $1 \leq i \leq n$, we hence obtain $\emptyset$-definable maps:

$$\begin{aligned} \partial_i^n : \text{Mor}(\mathcal{G}_n) &\to \text{Mor}(\mathcal{G}_{n-1}) \\ (\overline{a}, \overline{b})/E_n &\to (\overline{a}^{\wedge i}, \overline{b}^{\wedge i})/E_{n-1} \end{aligned}$$

and by setting $\partial_i^n(\pi(\overline{a})) = \pi(\overline{a}^{\wedge i})$, we can easily check that each ∂_i^n is an $\emptyset$-definable functor from $\mathcal{G}_n$ to $\mathcal{G}_{n-1}$. The Delta groupoid condition is immediate, so we constructed an $\emptyset$-type-definable Delta groupoid, using the ∂_i^n as the face maps. We denote it by $\Delta\mathcal{G}(q, \pi/\mathcal{P})$.

Returning to our first intuition, observe that these face maps have a clear interpretation as restrictions of partial automorphisms. Indeed, if $\overline{a} = (a_1, \cdots, a_n) \models r^{\otimes n}$ and $\overline{b} = (b_1, \cdots, b_n) \models r^{\otimes n}$, then an element σ of $\mathrm{Hom}(\mathcal{G}_n)(\pi(\overline{a}), \pi(\overline{b}))$ is a bijection:

$$\sigma : \mathrm{tp}(\overline{a}/\pi(\overline{a}))(\mathbb{M}) \to \mathrm{tp}(\overline{b}/\pi(\overline{b}))(\mathbb{M})$$

which is the restriction of an automorphism of $\mathbb{M}$ fixing $\mathcal{P}$ pointwise. The element $\partial_i^n(\sigma)$ of $\mathcal{G}_{n-1}$ is then the restriction of σ to a bijection:

$$\partial_i^n(\sigma) : \mathrm{tp}(\overline{a}^{\wedge i}/\pi(\overline{a}^{\wedge i}))(\mathbb{M}) \to \mathrm{tp}(\overline{b}^{\wedge i}/\pi(\overline{b}^{\wedge i}))(\mathbb{M})$$

which still is the restriction of the same global automorphism.

To summarize, we have obtained the following:

Theorem 3.3.19. *If (q, π) is relatively $\mathcal{P}$-internal, then for all n, the pair $(q^{\otimes n}, \pi^{\otimes n})$ is relatively $\mathcal{P}$-internal. Moreover, if we denote $\mathcal{G}_n$ the groupoid witnessing relative internality of $q^{\otimes n}$, then the $\mathcal{G}_n$ form a Delta groupoid $\Delta\mathcal{G}(q, \pi/\mathcal{P})$, using the natural restriction maps as face maps.*

For now, let us focus our attention on the binding groups of the fibers:

Notation. If $\overline{a} \models q^n$ for some n, then the type $\mathrm{tp}(\overline{a}/\pi(\overline{a}))$ is $\mathcal{P}$-internal, and we will denote $G_{\pi(\overline{a})}$ its binding group. It is $\mathrm{Mor}(\pi(\overline{a}), \pi(\overline{a}))$ in $\mathcal{G}_n$.

The Delta groupoid structure will, in particular, embed these groups into a projective system. More precisely, consider the directed system $\{\pi(\overline{a}) : \pi(\overline{a}) \models \pi^n(q^{\otimes n})$ for some $n\}$, with $(\pi(a_1), \cdots, \pi(a_n)) \leq (\pi(b_1), \cdots, \pi(b_m))$ if and only if $n \leq m$ and $\pi(a_i) = \pi(b_i)$ for all $i \leq n$. If $\pi(\overline{a}) \leq \pi(\overline{b})$, the restriction map $G_{\pi(\overline{b})} \to G_{\pi(\overline{a})}$ is definable, as it is a composition of face maps. These maps are then easily checked to form a projective system, and in particular give rise to the projective limit $\varprojlim G_{\pi(\overline{a})}$.

Note that the maps $G_{\pi(\overline{b})} \to G_{\pi(\overline{a})}$ are not necessarily surjective: it is possible that some element of $G_{\pi(\overline{a})}$ does not extend to a global automorphism fixing $\pi(\overline{b})$. However, this is the sole obstruction, and notably we can prove:

Remark 3.3.20. If $\pi(q)$ is $\mathcal{P}$-internal, then there is $m \in \mathbb{N}$ such that for all $k \geq n \geq m$, all $\pi(\overline{a}) \models q^{\otimes n}$, $\pi(\overline{b}) \models q^{\otimes k}$, and $\pi(\overline{a}) \leq \pi(\overline{b})$, the map $G_{\pi(\overline{b})} \to G_{\pi(\overline{a})}$ is surjective.

Proof. Let $\overline{a}_0 \models q^{\otimes m}$ be such that $\pi(\overline{a}_0)$ is a fundamental system of solutions for $\pi(q)$ (such an $\overline{a}_0$ exist by Remark 2.2.11). Then any m independent realizations of $\pi(q)$ will form a fundamental system of solutions. Hence for any $n \geq m$ and any $\overline{a} \models q^{\otimes n}$, the tuple $\pi(\overline{a})$ is a fundamental system of solutions for $\pi(q)$.

Fix $\overline{a} \models q^{\otimes n}$ for $n \geq m$ and $\pi(\overline{b}) \geq \pi(\overline{a})$, consider the map $G_{\pi(\overline{b})} \to G_{\pi(\overline{a})}$, we will now show it is surjective. So let $\sigma \in G_{\pi(\overline{a})}$, it is the restriction to $\mathrm{tp}(\overline{a}/\pi(\overline{a}))(\mathbb{M})$ of an automorphism $\tilde{\sigma}$ of $\mathbb{M}$. But $\pi(\overline{a})$ is a fundamental system of solutions for $\pi(q)$, and $\tilde{\sigma}$ fixes $\pi(\overline{a})$. Hence $\tilde{\sigma}$ fixes $\pi(q)(\mathbb{M})$, and in particular fixes $\pi(\overline{b})$. Therefore $\tilde{\sigma}$ restricts to an element of $G_{\pi(\overline{b})}$, and the image of this element under $G_{\pi(\overline{b})} \to G_{\pi(\overline{a})}$ has to be σ.

$\square$

It is now time to state the groupoid criterion for internality announced in the previous section. The relevant definitions will be:

Definition 3.3.21. The Delta groupoid $\Delta\mathcal{G}(q, \pi/\mathcal{P})$ is said to collapse if there is a tuple $\overline{a}$ of independent realizations of q such that for any $\overline{b} \geq \overline{a}$, the map $G_{\pi(\overline{b})} \to G_{\pi(\overline{a})}$ is injective. It is said to almost collapse if the maps $G_{\pi(\overline{b})} \to G_{\pi(\overline{a})}$ have finite kernel instead.

and we have:

Theorem 3.3.22. *The type q is internal (respectively almost internal) to $\mathcal{P}$ if and only if and only if the Delta groupoid $\mathcal{G}$ collapses (respectively almost collapses) and $\pi(q)$ is internal (respectively almost internal) to $\mathcal{P}$.*

Although this can be proved using groupoids, a much cleaner proof will be given in Chapter 4, after we introduce uniform relative internality.

Note that there is always a group morphism $\mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\pi(q)/\mathcal{P})$. If $\pi(q)$ is $\mathcal{P}$-internal, the target group is type-definable. Moreover, we have the following corollary of Theorem 3.3.22:

Corollary 3.3.23. *If the type q is internal to $\mathcal{P}$, then there is a definable (possibly over some extra parameters) short exact sequence:*

$$1 \to \varprojlim G_{\pi(\overline{a})} \to \mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\pi(q)/\mathcal{P}) \to 1$$

and the groups and morphisms are internal to $\mathcal{P}$.

Proof. Set $H = \ker(\mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\pi(q)/\mathcal{P}))$. Then we have a short exact sequence:

$$1 \to H \to \mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\pi(q)/\mathcal{P}) \to 1$$

Every group in this sequence is type-definable. Moreover, the left arrow is just inclusion, so is $\emptyset$-definable. As for the right arrow, if $\sigma \in \mathrm{Aut}(q/\mathcal{P})$ is represented by $(\overline{a}, \sigma(\overline{a}))$, we can simply send it to the class of $(\pi(\overline{a}), \pi(\sigma(\overline{a})))$, so the right arrow is definable. The groups and morphisms are internal to $\mathcal{P}$. So all we need to do to finish the proof is show that $\varprojlim G_{\pi(\overline{a})}$ is definably isomorphic to H.

Since q is $\mathcal{P}$-internal the Delta groupoid associated to q, π and $\mathcal{P}$ collapses. By Theorem 3.3.22 there is a tuple $\overline{b}$ of realizations of q such that $G_{\pi(\overline{c})} \to G_{\pi(\overline{b})}$ is injective for any $\pi(\overline{c}) \geq \pi(\overline{b})$. Moreover, since $\pi(q)$ is $\mathcal{P}$-internal, we can also assume, by Remark 3.3.20, that these maps are isomorphisms, hence $\varprojlim G_{\pi(\overline{a})} = G_{\pi(\overline{b})}$. By extending $\overline{b}$ we can assume both that $\overline{b}$ is a fundamental system of solutions for q and $\pi(\overline{b})$ is a fundamental system of solutions for $\pi(q)$. This allows us to define a morphism $G_{\pi(\bar{b})} \to \mathrm{Aut}(q/\mathcal{P})$ by sending $\sigma \in G_{\pi(b)}$ to the class of $(\bar{b}, \sigma(\bar{b}))$, it is well-defined because $\bar{b}$ is a fundamental system for q. This morphism is a relatively $\bar{b}$-definable map and it is injective, again because $\bar{b}$ is a fundamental system for q.

But $\pi(\bar{b})$ is a fundamental system for $\pi(q)$ and $\sigma \in G_{\pi(\bar{b})}$ so $\pi(\sigma(\bar{b})) = \pi(\bar{b})$. Hence the image of this map is contained in $H = \ker(\mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\pi(q)/\mathcal{P}))$. Finally, if $\sigma \in H$, then it has to fix $\pi(\bar{b})$, and hence restricts to an element of $G_{\pi(\bar{b})}$, which yields surjectivity of $G_{\pi(\bar{b})} \to H$.

$\square$

This short exact sequence carries some information about the type q. For example we have:

Proposition 3.3.24. *Suppose q is $\mathcal{P}$-internal and $\pi(q)$ is fundamental. If the short exact sequence:*

$$1 \to \varprojlim G_{\bar{a}} \to \mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\pi(q)/\mathcal{P}) \to 1$$

is definably split and $\mathcal{G}_1$ is connected, then $\mathcal{G}_1$ is retractable.

Proof. Since $\pi(q)$ is fundamental, an element of $\mathrm{Aut}(\pi(q)/\mathcal{P})$ is defined as the E'-class of $(\pi(a), \pi(b))$, for $\pi(a)$ and $\pi(b)$ two realizations of $\pi(q)$. Let s be a section of the short exact sequence. We can then define $g_{\pi(a),\pi(b)} = s((\pi(a), \pi(b))/E')$. This is uniformly $\emptyset$-definable, and the compatibility condition is easily checked.

$\square$

An interesting corollary is the following:

Corollary 3.3.25. *Suppose q is $\mathcal{P}$-internal and $\pi(q)$ is fundamental. If the short exact sequence:*

$$1 \to \varprojlim G_{\bar{a}} \to \mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\pi(q)/\mathcal{P}) \to 1$$

is definably split and $\mathcal{G}_1$ is connected, then the type q is the product of two weakly orthogonal $\mathcal{P}$-internal types.

Proof. Immediate consequence of Theorem 3.3.5 and Proposition 3.3.24.

□

Note that we obtained a converse to Theorem 3.3.12:

Theorem 3.3.26. *Suppose q is $\mathcal{P}$-internal. Assume $\mathcal{G}_1$ is connected, and $\pi(q)$ is fundamental. Then $\mathcal{G}_1$ is retractable if and only if the short exact sequence:*

$$1 \to \varprojlim G_{\overline{a}} \to \mathrm{Aut}(q/\mathcal{P}) \to \mathrm{Aut}(\pi(q)/\mathcal{P}) \to 1$$

is definably split.

Corollary 3.3.25 can be used to derive information about families of internal types in $\mathbb{M}$ from the behavior of definable group extensions in $\mathcal{P}$.

As an example of such an application, we will briefly mention Jin and Moosa's recent paper [18]. In said paper, the authors consider a type q in DCF_0, defined by an equation $\{x' = f(x)\}$, where f is a rational function, with parameters in any differential field. They then ask the following question:

Question 3.3.27. *Assuming q is $\mathcal{C}$-internal, when is the pullback $\delta\log^{-1}(q)$ by the logarithmic derivative also internal to the constants?*

Among other results, the authors obtain the following:

Theorem 3.3.28 (Jin-Moosa)**.** *If the binding group of q is not of dimension 3, then $\delta\log^{-1}(q)$ is internal to the constants if and only if it is bialgebraic with a product of two weakly orthogonal almost $\mathcal{C}$-internal types.*

Their proof goes through different cases, depending on the binding group of q. Our work allows for some simplification in the cases where this group is one dimensional. First remark that because the binding group of q is one dimensional, we know from results in Jin and Moosa's paper (Corollary 3.5) that q is weakly orthogonal to $\mathcal{C}$, and in particular the groupoid associated to $(\delta\log^{-1}(q), \delta\log)$ is connected. We give a rough idea of how to obtain some of their results using groupoids below.

Since q is $\mathcal{C}$-internal, Corollary 3.3.23 yields a short exact sequence:

$$1 \to \varprojlim G_{\overline{a}} \to \mathrm{Aut}(\delta \log^{-1}(q)/\mathcal{C}) \to \mathrm{Aut}(q/\mathcal{C}) \to 1$$

where:

1. $\mathrm{Aut}(q/\mathcal{C})$ is either $\mathbb{G}_a(\mathcal{C})$ or $\mathbb{G}_m(\mathcal{C})$, as it is a one dimensional connected affine algebraic group (in [18], Jin and Moosa prove that it cannot be an elliptic curve)
2. each $G_{\overline{a}}$ is an algebraic subgroup of $\mathbb{G}_m^n$, for some n
3. The projective limit $\varprojlim G_{\overline{a}}$ is isomorphic to some $G_{\overline{a}}$'s

Thus every group in this sequence is an algebraic group, and the morphisms between them also are algebraic. This implies that this sequence is algebraically (and definably) split. Therefore Corollary 3.3.25 applies.

The extension of these methods to the case of q not one-dimensional could be an interesting area of investigation. Remark that Corollary 3.3.25 does not apply anymore, as q is not fundamental, so more work would be needed. If q is already algebraic over $\mathcal{C}$, the groupoid is not connected anymore, so our methods will not apply. In dimension higher than one, it might be possible to obtain results by considering products of q.

Although the type-definable Delta groupoid $\Delta\mathcal{G}(q, \pi, \mathcal{P})$ captures a lot of information about the type q, it fails to account for interaction between fibers of two non-independent realizations of $\pi(q)$. Hence, it seems desirable to refine our methods and construct an algebraic object witnessing these interactions.

The ideal candidate for this is a simplicial groupoid. After our warm-up with Delta groupoids, we will now define these.

Definition 3.3.29. The category Δ is the category with finite ordinals as objects, and order preserving functions as morphisms.

Note that the only difference with the category $\hat{\Delta}$ used for Delta groupoids is that we now allow any order preserving function, not just strictly order preserving. And similarly, we have a categorical definition for simplicial groupoids:

Definition 3.3.30. Given a category $\mathcal{C}$, a simplicial object in $\mathcal{C}$ is a functor $F : \hat{\Delta}^{op} \to \mathcal{C}$. A simplicial groupoid is a simplicial object in the category of groupoids.

We also have the following characterisation:

Proposition 3.3.31. *A simplicial groupoid is equivalent to the following data :*

1. *For every integer $n \in \mathbb{N}$, a groupoid $\mathcal{G}_n$*
2. *For every integer $n \in \mathbb{N}^*$, and every $i \in [[0, \cdots n]]$, a groupoid morphism (that is, a functor) $\partial_i^n : \mathcal{G}_n \to \mathcal{G}_{n-1}$, called a face map*
3. *For every integer $n \in \mathbb{N}^*$, and every $i \in [[0, \cdots n]]$, a groupoid morphism $\eta_i^n : \mathcal{G}_n \to \mathcal{G}_{n+1}$, called a degeneracy map*

subject to the following conditions, known as the simplicial identities, for each n where they make sense :

1. $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ *for all* $i < j$
2. $\eta_i \circ \eta_j = \eta_{j+1} \circ \eta_i$ *for all* $i \leq j$
3. $\partial_i \circ \eta_j = \eta_{j-1} \circ \partial_i$ *for all* $i < j$
4. $\partial_i \circ \eta_j = \mathrm{id}$ *for* $i = j$ *or* $i = j + 1$
5. $\partial_i \circ \eta_j = \eta_j \circ \partial_{i-1}$ *for all* $i > j + 1$

Recall that we are fixing $q \in S(\emptyset)$ and an $\emptyset$-definable map π such that (q, π) is relatively $\mathcal{P}$-internal. We will now construct the simplicial groupoid $\Sigma\mathcal{G}(q, \pi/\mathcal{P})$ associated to it. In contrast to the Delta groupoid previously constructed, the morphism sets will fail to be uniformly definable, thus this will not yield a type-definable groupoid, according to our definition.

Let us first exhibit a sequence of groupoids $(\tilde{\mathcal{G}}_n)_{n\in\mathbb{N}}$. For all n, we let $\mathrm{Ob}(\tilde{\mathcal{G}}_n) = \{(\pi(a_1), \cdots \pi(a_n)) : a_1, \cdots, a_n \models q\}$. For $\pi(\overline{a}) \in \mathrm{Ob}(\tilde{\mathcal{G}}_n)$, we define $F_{\pi(\overline{a})}$ as the type definable set $q(x_1) \cup \cdots \cup q(x_n) \cup \{\pi(x_i) = \pi(a_i), i = 1, \cdots, n\}$, i.e. the fiber over $\pi(\overline{a})$. For $\pi(\overline{a}), \pi(\overline{b}) \in \mathrm{Ob}(\tilde{\mathcal{G}}_n)$, we define $\mathrm{Mor}(\pi(\overline{a}), \pi(\overline{b}))$ as the set of elementary bijections from $F_{\pi(\overline{a})}$ to $F_{\pi(\bar{b})}$, induced by an automorphism of $\mathbb{M}$ fixing $\mathcal{P}(\mathbb{M}) \cup \{\pi(a_i), \pi(b_i), i = 1, \cdots, n\}$ pointwise. Note that $\mathrm{Mor}(\pi(\overline{a}), \pi(\bar{b})) \subset \prod_{i=1}^{n} \mathrm{Mor}(\pi(a_i), \pi(b_i)))$, and $\mathrm{Mor}(\pi(\overline{a}), \pi(\overline{a}))$ is a subgroup of $\prod_{i=1}^{n} \mathrm{Mor}(\pi(a_i), \pi(a_i)))$.

An extra issue, compared with the construction of the Delta groupoid, is that the type-definable sets $F_{\pi(\overline{a})}$ do not have a canonical extension to a complete type (like $q^{\otimes n}$ in the Delta case). This is why we will not be able to get uniform type-definability of $\mathrm{Mor}(\pi(\overline{a}), \pi(\bar{b}))$ for $\pi(\overline{a}), \pi(\bar{b}) \in \mathrm{Ob}(\tilde{\mathcal{G}}_n)$.

Nevertheless, for fixed $\pi(\overline{a}) = \pi(a_1, \cdots, a_n) \in \mathrm{Ob}(\tilde{\mathcal{G}}_n)$ and $\pi(\bar{b}) = \pi(b_1, \cdots, b_n) \in \mathrm{Ob}(\tilde{\mathcal{G}}_n)$, we will now prove that $\mathrm{Mor}(\pi(\overline{a}), \pi(\bar{b}))$ is type definable over $\pi(\overline{a}), \pi(\bar{b})$. First remark that if $\mathrm{tp}(\pi(\overline{a})/\emptyset) \neq \mathrm{tp}(\pi(\bar{b})/\emptyset)$, then $\mathrm{Mor}(\pi(\overline{a}), \pi(\bar{b})) = \emptyset$. Thus we can assume $\mathrm{tp}(\pi(\overline{a})) = \mathrm{tp}(\pi(\bar{b}))$. Pick $(\overline{a}_1, \cdots, \overline{a}_n) = \alpha$ and $(\bar{b}_1, \cdots, \bar{b}_n) = \beta$, where each $\overline{a}_i$ (resp. $\bar{b}_i$) is a fundamental systems of solutions for $\mathrm{tp}(a_i/\pi(a_i))$ (resp. $\mathrm{tp}(b_i/\pi(b_i))$). As we assumed that $\mathrm{tp}(\pi(\overline{a})/\emptyset) = \mathrm{tp}(\pi(\bar{b})/\emptyset)$, we can pick these fundamental systems so that $\mathrm{tp}(\alpha/\emptyset) = \mathrm{tp}(\beta/\emptyset)$.

Let $r = \mathrm{tp}(\alpha/\emptyset) = \mathrm{tp}(\beta/\emptyset)$, note that we can apply π, coordinate-wise, to any realization of r, and the resulting pair (r, π) is relatively $\mathcal{P}$-internal. Thus, by our previous results, we obtain a type-definable groupoid $\mathcal{G}(r, \pi)$, and the set of morphisms $\mathrm{Mor}(\pi(\alpha), \pi(\beta))$, being in $\mathrm{Mor}(\mathcal{G}(r, \pi))$, is type definable over $\pi(\alpha), \pi(\beta)$, and thus over $\pi(\overline{a}), \pi(\bar{b})$.

Since $F_{\pi(\overline{a})} \subset \mathrm{dcl}(\alpha, \mathcal{P})$ and $F_{\pi(\bar{b})} \subset \mathrm{dcl}(\beta, \mathcal{P})$, the sets $\mathrm{Mor}(\pi(\overline{a}), \pi(\bar{b}))$ and $\mathrm{Mor}(\pi(\alpha), \pi(\beta))$ are in bijection. Thus we have proven that the sets of morphisms in the $\tilde{\mathcal{G}}_n$ are type-definable. Note that this also shows that the set obtained does not depend of the choice of the a_i and the type r. We now just have to prove that composition is definable.

Let us fix $\pi(\overline{a}), \pi(\bar{b}), \pi(\overline{c}) \in \mathrm{Ob}(\tilde{\mathcal{G}}_n)$, for some n. We can again assume that $\mathrm{tp}(\pi(\overline{a})) = \mathrm{tp}(\pi(\bar{b})) = \mathrm{tp}(\pi(\overline{c}))$, and pick fundamental systems α, β, γ having the same type r over $\emptyset$. As was argued previously, we see that composition in $\mathcal{G}(r, \pi)$

completely describes composition between $\mathrm{Mor}(\pi(\overline{a}), \pi(\overline{b}))$ and $\mathrm{Mor}(\pi(\overline{b}), \pi(\overline{c}))$ in $\tilde{\mathcal{G}}_n$. Thus the latter is definable.

Thus for each n we have obtained a groupoid $\tilde{\mathcal{G}}_n$. Its set of objects is type-definable, sets of morphisms are type-definable, but not uniformly, and composition is definable, but again, not uniformly.

To obtain a simplicial groupoid, we must now construct maps between these groupoids. We can pick the same face maps as for the Delta groupoid, i.e. restrictions.

As for the degeneracy maps, if $\overline{a} = (a_1, \cdots, a_n)$ and $\overline{b} = (b_1, \cdots, b_n)$, we let $\eta_i(\overline{a}) = (a_1, \cdots, a_i, a_i, \cdots, a_n)$ and $\eta_i(\overline{b}) = (b_1, \cdots, b_i, b_i, \cdots, b_n)$, i.e. we repeat the i-th coordinate. If $\sigma \in \mathrm{Mor}(\pi(\overline{a}), \pi(\overline{b}))$, we can then define $\eta_i(\sigma)$ in the same fashion, repeating the action of σ on the i-th fiber.

It is easy to check that these groupoids and maps form a simplicial groupoid, and that the face and degeneracy maps are definable.

To conclude this section, we will examine an example, pointed out to us by Rahim Moosa, showing that the morphism sets are not uniformly type-definable.

Example 3.3.32. We work in a monster model $\mathbb{M}$ of DCF_0. Consider the logarithmic derivative $\delta\log(x) = \frac{\delta(x)}{x}$, and the generic type q of $\{x, \delta(\delta\log(x)) = 0\}$. This is a well-know 2-analysable type over the constants $\mathcal{C}$, but not $\mathcal{C}$-internal.

More precisely, for any $a \models q$, the type $\mathrm{tp}(a/\delta\log(a))$ is stationary and $\mathcal{C}$-internal, so $(q, \delta\log)$ is relatively $\mathcal{C}$-internal. However, the groupoid $\tilde{\mathcal{G}}_2$ is not type-definable.

Proof. This is mostly well known, and we only need to prove that $\tilde{\mathcal{G}}_2$ is not type-definable. We refer the reader to [4] for an elegant proof of the rest of this statement.

Consider two generic constants a and b. Then the binding groups of $\delta\log^{-1}(a)$ and $\delta\log^{-1}(b)$ are isomorphic to $\mathbb{G}_m(\mathcal{C})$, the multiplicative group of the constants, and are definable over a and b, respectively.

We make the assumption that $\tilde{\mathcal{G}}_2$ is type-definable, and will derive a contradiction. In particular it implies that the groups G_{ab} are uniformly type definable over the parameters a, b, as they are $\mathrm{Mor}((a,b),(a,b))$ in $\tilde{\mathcal{G}}_2$. In fact, as DCF_0 is ω-stable, these groups are uniformly definable.

Consider the map $G_{a,b} \to G_b$, its kernel consists of elements of $G_a \times G_b$, such that the G_b coordinate is the identity. Thus it is of the form $K_a \times \{1\}$, where K_a is a subgroup of G_a. Since this face map is $\{a, b\}$-definable, the group K_a is definable as well, and by strong minimality, it is either finite or equal to G_a.

The set $\{b \models \delta\log(q), \ker(G_{a,b} \to G_b) = \{G_a \times \{1\}\}$ is therefore relatively definable over a, that is, it is the intersection of $\delta\log(q)$ with an a-definable set. Hence the set $\{b \models \delta\log(q), \ker(G_{a,b} \to G_b) \neq \{1\} \times G_a\}$ is also relatively a-definable, and by the discussion above, we have $\ker(G_{a,b} \to G_b) \neq G_a \times \{1\}$ if and only if $\ker(G_{a,b} \to G_b) = K_a \times \{1\}$, with $K_a < G_a$ finite. In turns, this is equivalent to $\delta\log^{-1}(a)(\mathbb{M}) \subset \mathrm{acl}(\delta\log^{-1}(b) \cup \mathcal{C})$. In conclusion, the set $X_a = \{b \models \delta\log(q), \delta\log^{-1}(a)(\mathbb{M}) \subset \mathrm{acl}(\delta\log^{-1}(b) \cup \mathcal{C})\}$ is relatively a-definable.

We will obtain a contradiction from this relative definability. By Lemma 2.4.20, we see that $\delta\log^{-1}(a)(\mathbb{M}) \subset \mathrm{acl}(\delta\log^{-1}(b) \cup \mathcal{C})$ if and only if for all $\alpha \in \delta\log^{-1}(a)(\mathbb{M})$ there is $n > 0$ such that $\alpha^n \in F$, where F is the differential field generated by $\mathcal{C}$ and $\delta\log^{-1}(b)$.

Let $\beta \in \delta\log^{-1}(b)$, then $\delta\log^{-1}(b) = \{c \cdot \beta : c \in \mathcal{C}\}$, and $\delta(\beta) = b \cdot \beta$. Hence $F = \mathcal{C} < \beta >$, the field generated by β and $\mathcal{C}$.

Therefore, for $\alpha \in \delta\log^{-1}(a)$, we have that $\alpha^n \in F$ if and only if there are polynomials $P, Q \in \mathcal{C}[X]$ such that $\alpha^n = \frac{P(\beta)}{Q(\beta)}$. Set $P = \sum_{i=1}^{k} p_i X^i$ and $Q = \sum_{j=1}^{l} q_j X^j$.

Hence $\alpha^n Q(\beta) = P(\beta)$, and applying δ we get:

$$\begin{aligned} P(\beta)' = \delta(\alpha^n Q(\beta)) &= an\alpha^n Q(\beta) + \alpha^n Q(\beta)' \\ &= anP(\beta) + \alpha^n Q(\beta)' \end{aligned}$$

Combining these two equations and identifying α^n yields:

$$\frac{P(\beta)}{Q(\beta)} = \frac{P(\beta)' - anP(\beta)}{Q(\beta)'}$$

hence:

$$P(\beta)Q(\beta)' = Q(\beta)(P(\beta)' - an(P(\beta)))$$

A straightforward computation gives us $P(\beta)' = \sum_{i=1}^{k} ibp_i\beta^i$ and $Q(\beta)' = \sum_{j=1}^{l} jbq_j\beta^j$.

Since P and Q have coefficients in $\mathcal{C}$, the previous identity is a polynomial equation over $\mathcal{C}$ satisfied by β. We can pick β to be a generic point of $\delta\log^{-1}(\mathcal{C})$, implying it cannot satisfy such an equation, unless all the coefficients are zero.

In particular, the dominant coefficients are zero, yielding:

$$p_k lbq_l = q_l(kbp_k - anp_k)$$

which simplifies into:

$$(k - l)b = na$$

If $k = l$, this implies that $a = 0$ (as $n > 0$), but again, this cannot happen because a is generic in $\mathcal{C}$. Hence $k \neq l$, and therefore $b = \frac{n}{k-l}a$. So $b \in X_a$ implies that there is $r \in \mathbb{Q} \setminus \{0\}$ such that $b = r \cdot a$.

Reciprocally, suppose that there are non zero $n, m \in \mathbb{Z}$ such that $b = \frac{n}{m}a$, and let $\beta \in \delta\log^{-1}(b)$. Consider some γ such that $\gamma^n = \beta^m$. Then we have $\delta(\gamma^n) = n\delta(\gamma)\gamma^{n-1}$, but also:

$$\begin{aligned}
\delta(\gamma^n) &= \delta(\beta^m) \\
&= mb\beta^m \\
&= na\beta^m \\
&= na\gamma^n
\end{aligned}$$

hence we get $\delta(\gamma) = a\gamma$. Therefore $\gamma \in \delta\log^{-1}(a)$. But $\gamma \in \mathrm{acl}(\beta)$, which implies that $\delta\log^{-1}(a)(\mathbb{M})$ is a subset of $\mathrm{acl}(\delta\log^{-1}(b) \cup \mathcal{C})$, yielding in turns that $b \in X_a$.

To sum up, we have proved that $b \in X_a$ if and only if there is $r \in \mathbb{Q}$ such that $b = r \cdot a$. The set X_a being a-type definable, this would imply that $\mathbb{Q}$ is a-type definable in $\mathcal{C}$, which is a contradiction. Hence the groupoid $\tilde{G}_2$ cannot be type definable.

$\square$

Note that uniform definability of the morphism sets is the only obstruction to these groupoids being type-definable. In fact, the simplicial groupoid $\sum \mathcal{G}(q, \pi/\mathcal{P})$ still contains a lot of information about q, in the form of type-definable groups. More precisely, the binding groups $G_{\pi(\overline{a})} = \mathrm{Aut}(\mathrm{tp}(\overline{a}/\pi(\overline{a})))$, where $\overline{a}$ is a sequence of (not necessarily independent) realizations of q. A deeper study of these simplicial groupoids is a promising avenue for future research.

CHAPTER 4

UNIFORM RELATIVE INTERNALITY

This chapter is joint with my advisor Anand Pillay and Rémi Jaoui.

We will introduce the new notion of uniform relative internality. It will turn out to be equivalent to the previously defined collapse of a Delta groupoid. In fact, the first use of uniform relative internality was to simplify the proof of the Delta groupoid internality criterion given by Theorem 3.3.22.

Indeed, in most concrete applications, uniform relative internality is easier to use than Delta groupoids. This is why, starting from now, it will become the focal point of our investigation, while groupoids fade in the background.

4.1 First results, link with Delta groupoids

Recall that $\mathcal{P}$ is a family of types over $\emptyset$, and that we assume $\emptyset = \mathrm{acl}(\emptyset)$. Let us start, as is now expected, with a definition:

Definition 4.1.1. Let (q, π) be relatively $\mathcal{P}$-internal, with $q \in S(\emptyset)$. Then (q, π) is said to be uniformly relatively $\mathcal{P}$-internal (resp. almost $\mathcal{P}$-internal) if there is a tuple e such that for any $a \models q$, we have $a \in \mathrm{dcl}(\pi(a), e, \mathcal{P})$ (resp. $a \in \mathrm{acl}(\pi(a), e, \mathcal{P})$).

This definition would make sense replacing $\emptyset$ by any algebraically closed set of parameters A. The choice of $\emptyset$ here is only a matter of convenience. However, as we will see in Section 4.4, different choice for A will lead to different behavior of uniformly internal types.

Even more important is that the family $\mathcal{P}$ is over a *fixed* set of parameters. Indeed, suppose that we replace $\mathcal{P}$ with the family $\mathcal{Q}$ of all types, over any set of parameters, that are $\mathcal{P}$-internal. Then any relatively $\mathcal{P}$-internal pair is uniformly $\mathcal{Q}$-internal.

The chosen name is quite transparent: all the types $\operatorname{tp}(a/\pi(a))$ are $\mathcal{P}$-internal, and we can choose a tuple witnessing internality uniformly for all $\pi(a)$ realizing $\pi(q)$.

The simplest way for (q, π) to be relatively $\mathcal{P}$-internal is for it to decompose as $q = \pi(q) \otimes r$, where r is a $\mathcal{P}$-internal type.

Definition 4.1.2. The pair (q, π) is said to be trivial (resp. almost) if for any $a \models q$ there is $c \in \operatorname{dcl}(a)$, independent from $\pi(a)$ over $\emptyset$, such that $a \in \operatorname{dcl}(c, \pi(a))$ (resp. $\operatorname{acl}(c, \pi(a))$) and $\operatorname{tp}(c/\emptyset)$ is $\mathcal{P}$-internal. In that case, if $r = \operatorname{tp}(c)$, we have $q = r \otimes \pi(q)$ (resp. q is in the algebraic closure of $r \otimes \pi(q)(\mathbb{M})$).

In practice, we will often consider a $\mathcal{P}$-internal type $\operatorname{tp}(a/b)$, and consider the type $\operatorname{tp}(ab/b)$. If π is the projection on the b coordinate, then $(\operatorname{tp}(ab), \pi)$ is relatively $\mathcal{P}$-internal. To make notation less cumbersome, we introduce the following:

Definition 4.1.3. The (almost) $\mathcal{P}$-internal type $\operatorname{tp}(a/b)$ is said to be (almost) uniformly $\mathcal{P}$-internal if $\operatorname{tp}(ab)$ is uniformly relatively (almost) $\mathcal{P}$-internal via the projection on the b coordinate.

Remark 4.1.4. If (q, π) is uniformly relatively internal, then q is internal to the family of types $\mathcal{P} \cup \{\pi(q)\}$. However, the converse is false.

For example, consider q orthogonal to $\mathcal{P}$ and $\pi : q^{(2)} \to q$ the projection on the second coordinate. Then $q^{(2)}$ is internal to $\mathcal{P} \cup \{\pi(q^{(2)})\}$. However, the pair $(q^{(2)}, \pi)$ is not uniformly relatively internal to $\mathcal{P}$. Indeed, this would mean that there is a tuple e such that for any $(a, b) \models q^{(2)}$, we have $a \in \operatorname{dcl}(b, e, \mathcal{P})$. Picking (a, b) independent of e, we see that this contradicts orthogonality to $\mathcal{P}$.

Note the similarity with how internal types were defined: the only difference is the need to introduce $\pi(a)$. In practice, this means that the same techniques used for working with internal types can be used to work with uniformly relatively internal types. As an illustration, let us prove:

Proposition 4.1.5. *Let (q, π) be uniformly relatively almost $\mathcal{P}$-internal. Then there is a Morley sequence $(a_i)_{i=1\cdots n}$ of realizations of q such that for any $a \models q$, independent of $(a_i)_{i=1\cdots n}$, we have $a \in \operatorname{acl}(\pi(a), a_1, \cdots, a_n, \mathcal{P})$.*

Proof. Let e be such that for all $a \models q$ we have $a \in \operatorname{acl}(\pi(a), e, \mathcal{P})$, which exists by assumption. Let $a \models q$ be independent from e over the empty set, and let $c \in \mathcal{P}$ be such that $a \in \operatorname{acl}(\pi(a), e, c)$.

Consider $\operatorname{tp}(ac/\operatorname{acl}(e))$, it is a stationary type, let d be its canonical base. Pick $(a_i c_i)_{i\in\mathbb{N}}$, a Morley sequence in $\operatorname{tp}(ac/\operatorname{acl}(e))$, which we can assume to be independent from ac over e. We know that $ac \mathop{\smile\!\!\!\!\!\mid}_d \operatorname{acl}(e)$, and from this and the assumption, forking calculus yields $a \in \operatorname{acl}(\pi(a), c, d)$. But $d \in \operatorname{acl}((a_i c_i)_{1\leq i\leq n})$ for some n, hence $a \in \operatorname{acl}(\pi(a), (a_i c_i)_{1\leq i\leq n}, c)$, so $a \in \operatorname{acl}(\pi(a), (a_i)_{1\leq i\leq n}, \mathcal{P})$.

Now let a' be any realization of q independent from $(a_i)_{1\leq i\leq n}$ over the empty set. Since a is independent from e over the empty set, and independent over e of the sequence $(a_i)_{i\in\mathbb{N}}$, we have that a is independent from $(a_i)_{i\in\mathbb{N}}$ over the empty set. Since $q = \operatorname{tp}(a/\emptyset)$ is stationary, this implies that $\operatorname{tp}(a/(a_i)_{i\in\mathbb{N}}) = \operatorname{tp}(a'/(a_i)_{i\in\mathbb{N}})$, hence $a' \in \operatorname{acl}(\pi(a'), (a_i)_{1\leq i\leq n}, \mathcal{P})$.

□

In some cases, the independence restriction of Proposition 4.1.5 is not required. We first recall:

Definition 4.1.6. The preweight of a type $p = \operatorname{tp}(a/A)$, denoted $\operatorname{prwt}(p)$, is the largest cardinal κ such that there exists an A-independent sequence $(b_i)_{i<\kappa}$ such that $a \not\mathop{\smile\!\!\!\!\!\mid}_A b_i$ for all i.

The weight of p, denoted $\operatorname{wt}(p)$, is $\sup(\{\operatorname{prwt}(q), q \text{ non-forking extension of } p\})$.

Proposition 4.1.7. *Let (q, π) be uniformly relatively almost $\mathcal{P}$-internal, and assume that $\mathrm{wt}(q)$ is finite. Then there is a Morley sequence $(a_k)_{k=1\cdots r}$ of realizations of q such that for any $a \models q$, we have $a \in \mathrm{acl}(\pi(a), a_1, \cdots, a_n, \mathcal{P})$.*

Proof. This is essentially the same proof as 4.1.5. However, once we obtain the sequence $(a_i)_{i=1\cdots n} = \overline{a}$, we consider a Morley sequence $(\overline{a}_j)_{j\in\mathbb{N}}$ in $\mathrm{tp}(\overline{a})$.

Because q has finite weight, there is $m \in \mathbb{N}$ such that for any realization a' of q, there is $1 \leq j \leq m$ such that $a' \not\perp \overline{a}_j$. By mimicking the proof of 4.1.5, we see that $(\overline{a}_j)_{j=1\cdots m}$ is the required Morley sequence.

□

In particular, this is valid for any type q of a superstable theory, as any type then have finite weight. One last, very useful consequence of these methods is:

Proposition 4.1.8. *The type q is uniformly $\mathcal{P}$-internal (resp. almost) via π if and only if for some (any) $a \models q$, there is a tuple e, independent from a over $\emptyset$, such that $a \in \mathrm{dcl}(\pi(a), e, \mathcal{P})$ (reps. $a \in \mathrm{acl}(\pi(a), e, \mathcal{P})$).*

Proof. The left to right direction is immediate. For the other direction, one simply has to copy the proof of Proposition 4.1.5.

□

The next proposition allows, if one does not mind replacing uniform internality by almost uniform internality, to choose parameters internal to $\mathcal{P}$:

Proposition 4.1.9. *Suppose (q, π) is uniformly relatively almost $\mathcal{P}$-internal. Then there is a tuple of parameters t such that:*

1. *for any $a \models q$ independent from t over $\emptyset$, we have $a \in \mathrm{acl}(t, \pi(a), \mathcal{P})$*
2. *$\mathrm{tp}(t/\emptyset)$ is $\mathcal{P}$-internal*

Proof. By assumption, there are $a \models q$, a tuple e, independent from a over $\emptyset$, and $c \in \mathcal{P}$, such that $a \in \mathrm{acl}(\pi(a), e, c)$. Consider $t = \mathrm{Cb}(\mathrm{stp}(ac/e))$.

By properties of canonical bases, we have $ac \mathop{\smile\!\!\!\!\!\!\;|}_t e$, thus $a \in \operatorname{acl}(t, \pi(a), c)$. As $t \in \operatorname{acl}(e)$, we also know that a is independent from t over $\emptyset$. As q is stationary, for any $b \models q$ independent from t over $\emptyset$, we have $b \in \operatorname{acl}(t, \pi(b), \mathcal{P})$. This yields property 1.

For the second property, recall that $t \in \operatorname{dcl}((a_i c_i)_{i=1\cdots n})$, for $(a_i c_i)_{i=1\cdots n}$ a Morley sequence in $\operatorname{stp}(ac/e)$. Because $a \mathop{\smile\!\!\!\!\!\!\;|} e$, we can prove, by induction and forking calculus, that $a_1 \cdots a_n \mathop{\smile\!\!\!\!\!\!\;|} e$, and therefore $a_1 \cdots a_n \mathop{\smile\!\!\!\!\!\!\;|} t$. Lastly, since $\operatorname{tp}(c_i) = \operatorname{tp}(c)$ for all i, we see that $c_i \in \mathcal{P}$ for all i. This, combined with the independence previously obtained, yields almost $\mathcal{P}$-internality of $\operatorname{tp}(t/\emptyset)$.

To replace t by an internal tuple, recall the following general fact: if $\operatorname{tp}(t/\emptyset)$ is almost $\mathcal{P}$-internal, there is $d \in \operatorname{dcl}(d)$ such that $\operatorname{tp}(d/\emptyset)$ is $\mathcal{P}$-internal and $t \in \operatorname{acl}(d)$. Applying this to the previously obtained t, we get our result.

$\square$

This is a clarifying structural result. Indeed, consider the type $\widetilde{q} = \operatorname{tp}(t, \pi(a))$, the pair $(\widetilde{q}, \rho)$ is relatively $\mathcal{P}$-internal, with ρ projection on the $\pi(a)$ coordinate. In fact, it is trivial, as $\widetilde{q} = r \otimes \pi(q)$, where $r = \operatorname{tp}(t/\emptyset)$. As we could prove a similar result for uniform almost $\mathcal{P}$-internality, we have proved:

Theorem 4.1.10. *The pair (q, π) is uniformly relatively almost $\mathcal{P}$-internal if and only if there exists a trivial pair $(\widetilde{q}, \widetilde{\pi})$ such that $q(\mathbb{M}) \in \operatorname{acl}(\widetilde{q}(\mathbb{M}), \mathcal{P})$.*

Another enlightening structural corollary, pointed out by Anand Pillay, is:

Theorem 4.1.11. *Let $\mathcal{P}_{\text{int}}$ be the internal closure of $\mathcal{P}$, that is, the family of partial types over $\emptyset$ that are internal to $\mathcal{P}$. The pair (q, π) is uniformly almost $\mathcal{P}$-internal if and only if for some (any) $a \models q$, the type $\operatorname{tp}(a/\pi(a))$ is $\mathcal{P}_{\text{int}}$-algebraic.*

Proof. The left to right direction is an immediate consequence of Proposition 4.1.9.

For the right to left direction, assume that $\operatorname{tp}(a/\pi(a))$ is $\mathcal{P}_{\text{int}}$-algebraic. So there is a tuple c of realizations of $\mathcal{P}_{\text{int}}$ such that $a \in \operatorname{acl}(c, \pi(a))$. By definition of $\mathcal{P}_{\text{int}}$, there is a tuple t such that $c \in \operatorname{acl}(t, \mathcal{P})$ and $c \downarrow t$. We can assume that $t \downarrow_c a$ by picking a realization of $\operatorname{tp}(t/c)|_a$.

This yields $t \downarrow a$ and $a \in \operatorname{acl}(\pi(a), t, \mathcal{P})$, hence (q, π) is uniformly almost $\mathcal{P}$-internal by Proposition 4.1.8.

□

Note that since the tuple t we obtained is $\mathcal{P}$-internal, it has, in particular, a binding group $\operatorname{Aut}(\operatorname{tp}(t/\emptyset)/\mathcal{P})$. Information on this group can be recovered from the binding groups of fibers:

Lemma 4.1.12. *Let t, a, c be as in 4.1.9, and $(a_i, c_i))$ the Morley sequence obtained in the proof. Then the binding group $\operatorname{Aut}(\operatorname{tp}(t/(\pi(a_i))_{i=1\cdots n})/\mathcal{P})$ is the image, under a definable finite-to-one morphism, of $\operatorname{Aut}(\operatorname{tp}(a_1, \cdots, a_n/(\pi(a_i))_{i=1\cdots n}))/\mathcal{P})$. Moreover, if $\pi(q)$ is orthogonal to $\mathcal{P}$, we have:*

$$\begin{aligned}\operatorname{Aut}(\operatorname{tp}(t/\emptyset)/\mathcal{P}) &= \operatorname{Aut}(\operatorname{tp}(t/(\pi(a_i))_{i=1\cdots n})/\mathcal{P}) \\ &= \operatorname{Aut}(\operatorname{tp}(a_1, \cdots, a_n/(\pi(a_i))_{i=1\cdots n})/\mathcal{P})\end{aligned}$$

Proof. For the first part, note that $t \in \operatorname{dcl}((a_i)_{i=1\cdots n}, \mathcal{P})$. This allows to construct a $\pi(a_1), \cdots, \pi(a_n)$-definable morphism:

$$\operatorname{Aut}(\operatorname{tp}(a_1, \cdots, a_n/(\pi(a_i))_{i=1\cdots n})/\mathcal{P}) \to \operatorname{Aut}(\operatorname{tp}(t/(\pi(a_i))_{i=1\cdots n})/\mathcal{P})$$

by taking $\sigma \in \operatorname{Aut}(\operatorname{tp}(a_1, \cdots, a_n/(\pi(a_i))_{i=1\cdots n})/\mathcal{P})$ to its unique extension as an automorphism in $\operatorname{Aut}(\operatorname{tp}(t/(\pi(a_i))_{i=1\cdots n})/\mathcal{P})$. Note that this extension exists by stability, and is unique because $t \in \operatorname{dcl}((a_i)_{i=1\cdots n}, \mathcal{P})$. It is finite-to-one because for each i we have $a_i \in \operatorname{acl}(t, \pi(a_i), \mathcal{P})$.

Now assume that $\pi(q)$ is orthogonal to $\mathcal{P}$, we need to show that the binding group of $\operatorname{tp}(t/\emptyset)$ is the same as the binding group of $\operatorname{tp}(t/\pi(a_1), \cdots, \pi(a_n))$.

Recall that a fundamental system for $\operatorname{tp}(t/\pi(a_1) \cdots \pi(a_n))$ is given by a Morley sequence $(t_i)_{i=1\cdots m}$ in $\mathrm{tp}(t/\pi(a_1)\cdots\pi(a_n))$. As $t \downarrow \pi(a_1), \cdots, \pi(a_n)$, we can prove, by induction and forking calculus, that such a sequence is also a Morley sequence in $\mathrm{tp}(t/\emptyset)$. Thus, if m is picked large enough, it is also a fundamental system of solutions for $\mathrm{tp}(t/\emptyset)$. Fix such a Morley sequence $(t_1, \cdots t_m)$. To ease notation, we will, for the rest of the proof, write $\overline{a} = a_1, \cdots, a_n$ and $\overline{t} = t_1, \cdots, t_m$.

A morphism $\sigma \in \mathrm{Aut}(\mathrm{tp}(t/\emptyset)/\mathcal{P})$ is given by the class of $(\overline{t}, \sigma(\overline{t}))$ under some $\emptyset$-definable equivalence relation. Showing that $\overline{t}$ and $\sigma(\overline{t})$ have the same type over $\mathcal{P} \cup \{\pi(a_1), \cdots, \pi(a_n)\}$ will also show, by Fact 2.2.14, that there is $\tau \in \mathrm{Aut}(\mathrm{tp}(t/\pi(\overline{a}))/\mathcal{P})$ taking $\overline{t}$ to $\sigma(\overline{t})$.

If we can show this for any $\sigma \in \mathrm{Aut}(\mathrm{tp}(t/\emptyset)/\mathcal{P})$, we would have shown the existence of a definable group homomorphism $\mathrm{Aut}(\mathrm{tp}(t/\emptyset)/\mathcal{P}) \to \mathrm{Aut}(\mathrm{tp}(t/\pi(\overline{a}))/\mathcal{P})$. This morphism would be injective by construction, and surjective because if $\overline{t} \equiv_{\mathcal{P},\pi(\overline{a})} \sigma(\overline{t})$, then also $\overline{t} \equiv_{\mathcal{P}} \sigma(\overline{t})$. Therefore, this would yield the desired group isomorphism.

So fix $\sigma \in \mathrm{Aut}(\mathrm{tp}(t/\emptyset)/\mathcal{P})$, we need to show that $\sigma(\overline{t})$ has the same type as $\overline{t}$ over $\mathcal{P} \cup \{\pi(\overline{a})\}$. As $\pi(q)$ is orthogonal to $\mathcal{P}$ and $\mathrm{tp}(t/\emptyset)$ is $\mathcal{P}$-internal, we deduce that $\pi(q)$ is orthogonal to the family of types $\mathcal{P} \cup \{\mathrm{tp}(t/\emptyset)\}$. Hence $\pi(q)^{\otimes n}$ is also orthogonal to this family.

In particular, as $\pi(q)^{\otimes n}$ is stationary, and both $\pi(\overline{a})$ and $\sigma(\pi(\overline{a}))$ realize $\pi(q)^{\otimes n}$, they have the same type over $\mathcal{P} \cup \mathrm{tp}(t/\emptyset)(\mathbb{M})$. Therefore, by Fact 2.2.14 again, there is an automorphism τ fixing $\mathcal{P} \cup \mathrm{tp}(t/\emptyset)(\mathbb{M})$ pointwise and taking $\pi(\overline{a})$ to $\sigma(\pi(\overline{a}))$.

Fix an extension of σ to $\mathrm{Aut}(\mathbb{M})$, which we will also denote σ (even though it is not unique). Consider the automorphism $\tau^{-1} \circ \sigma$, it fixes $\mathcal{P}$ pointwise, fixes $\pi(\overline{a})$, and takes $\overline{t}$ to $\sigma(\overline{t})$. This proves that $\overline{t} \equiv_{\mathcal{P},\pi(\overline{a})} \sigma(\overline{t})$, what we wanted to show.

□

One of the features of uniform relative internality is that it is the exact property needed to force an analysable type to be internal. Indeed we can prove:

Observation 4.1.13. *Suppose (q, π) is relatively $\mathcal{P}$-internal. Then q is $\mathcal{P}$-internal (resp. almost $\mathcal{P}$-internal) if and only if $\pi(q)$ is $\mathcal{P}$-internal (resp. almost $\mathcal{P}$-internal) and (q, π) is uniformly relatively $\mathcal{P}$-internal (resp. uniformly relatively almost $\mathcal{P}$-internal).*

Proof. We will only treat the case of internality and uniform internality, as the almost internal case is similar.

Suppose first that q is internal to $\mathcal{P}$. We immediately get that $\pi(q)$ is internal as well. It also yields a fundamental system of solutions, denote it $\overline{a}$, which we can pick as a tuple of independent realizations of q. If we now pick any $b \models q$, we have $b \in \operatorname{dcl}(\overline{a}, \mathcal{P})$, hence also $b \in \operatorname{dcl}(\overline{a}, \pi(b), \mathcal{P})$.

For the other implication, assume that (q, π) is uniformly relatively $\mathcal{P}$-internal and $\pi(q)$ is $\mathcal{P}$-internal. Hence, the type q is internal to the family of types $\mathcal{P} \cup \{\pi(q)\}$. But because $\pi(q)$ is $\mathcal{P}$-internal, this implies that q itself is $\mathcal{P}$-internal (see [29], Remark 7.4.3).

□

We now briefly return to groupoids in order to prove Theorem 3.3.22. The key observation is that uniform relative internality corresponds exactly to collapse of the Delta groupoid.

Theorem 4.1.14. *Let (q, π) be relatively $\mathcal{P}$-internal. Then (q, π) is uniformly relatively $\mathcal{P}$-internal (resp. almost) if and only if the Delta groupoid $\Delta\mathcal{G}(q, \pi/\mathcal{P})$ collapses (resp. almost).*

Proof. Again, we will only prove the equivalence of collapsing of the groupoid and uniform $\mathcal{P}$-internality, the other equivalence being proved in a similar way. Suppose first that q is uniformly relatively $\mathcal{P}$-internal. By Proposition 4.1.5, there is a tuple $\overline{a}$ of independent realizations of q such that for any $b \models q$ independent of $\overline{a}$, we have $b \in \operatorname{dcl}(\overline{a}, \pi(b), \mathcal{P})$.

Pick any $\pi(\bar{b}) > \pi(\bar{a})$, it is enough to prove that the map $G_{\pi(\bar{b})} \to G_{\pi(\bar{a})}$ is injective. Note that $\text{tp}(\bar{a}/\pi(\bar{a}))$ is $\mathcal{P}$-internal, and thus has a fundamental system of solutions $\bar{a}_0$. The type $\text{tp}(\bar{b}/\pi(\bar{b}))$ is also $\mathcal{P}$-internal, hence also has a fundamental system of solutions $(b_1, \cdots, b_n)$. For each i, either b_i is in $\pi^{-1}(\pi(\bar{a}))$, and hence in $\text{dcl}(\bar{a}_0, \mathcal{P})$, or $\pi(b_i)$ is independent of $\pi(\bar{a})$ over $\emptyset$, so b_i is independent of $\bar{a}$ over $\emptyset$. In this second case, the assumption yields $b_i \in \text{dcl}(\bar{a}, \pi(b_i), \mathcal{P}) \subset \text{dcl}(\bar{a}_0, \pi(b_i), \mathcal{P})$. Hence we obtain $b_i \in \text{dcl}(\bar{a}_0, \pi(b_i), \mathcal{P})$ for all i, so $\text{tp}(\bar{b}/\pi(\bar{b}))(\mathbb{M}) \subset \text{dcl}(\bar{a}_0, \pi(\bar{b}), \mathcal{P})$.

Now let $\sigma \in G_{\pi(\bar{b})}$ be such that its image under $G_{\pi(\bar{b})} \to G_{\pi(\bar{a})}$ is the identity. Then it has to fix $\bar{a}_0$, and it fixes $\pi(\bar{b})$ and $\mathcal{P}$ too. Since we just proved $\text{tp}(\bar{b}/\pi(\bar{b}))(\mathbb{M}) \subset \text{dcl}(\bar{a}_0, \pi(\bar{b}), \mathcal{P})$, this implies that has to fix $\text{tp}(\bar{b}/\pi(\bar{b}))(\mathbb{M})$ pointwise, so it is the identity of $G_{\pi(\bar{b})}$.

For the other implication, suppose that $\Delta\mathcal{G}(q, \pi/\mathcal{P})$ collapses. Hence there is a tuple $\bar{a}$ of independent realizations of q such that for any $\pi(\bar{b}) \geq \pi(\bar{a})$, the map $G_{\pi(\bar{b})} \to G_{\pi(\bar{a})}$ is injective. The type $\text{tp}(\bar{a}/\pi(\bar{a}))$ is $\mathcal{P}$-internal, and it has a fundamental system of solutions. From now on, we replace $\bar{a}$ by this fundamental system.

We need to prove that for any $b \models q$ independent of $\bar{a}$, we have $b \in \text{dcl}(\bar{a}, \pi(b), \mathcal{P})$. To do so, it is enough, by Fact 2.2.14, to prove that any automorphism σ of $\mathbb{M}$ fixing $\bar{a}, \pi(b)$ and $\mathcal{P}$ pointwise has to fix b. So consider such an automorphism σ. It restricts to $\sigma \in G_{\pi(b)\pi(\bar{a})}$, as it fixes $\pi(b)$ and $\pi(\bar{a})$. But it also fixes $\bar{a}$, which is a fundamental system of solutions for $\text{tp}(\bar{a}/\pi(\bar{a}))$. Hence, its image under the map $G_{\pi(\bar{a})\pi(b)} \to G_{\pi(\bar{a})}$ is the identity, so by collapse assumption, it is itself the identity in $G_{\pi(\bar{a})\pi(b)}$, and in particular fixes b.

□

The internality criterion of Theorem 3.3.22 is now immediate: assuming $\pi(q)$ is internal, the type q is internal if and only if it is uniformly relatively internal if and only if the associated Delta groupoid collapses.

Recall that one way to obtain a relatively internal type is to consider an internal type $\operatorname{tp}(a/b)$. In that case, the pair $(\operatorname{tp}(ab/\emptyset), \pi)$ is relatively $\mathcal{P}$-internal, with π the projection on the b-coordinate. Thus we moved from a single internal type to a family of internal types. This is where this dissertation got its name from: all our work can be seen as the development of tools for the study of families of internal types.

By considering the behavior of the relatively internal type $\operatorname{tp}(ab)$, it is possible to refine our understanding of internality. Indeed, not all internal types are created equal, and $\operatorname{tp}(ab/\emptyset)$ can exhibit various behaviors. The first refining of internality we will consider was first defined by Moosa in [24]:

Definition 4.2.1. The almost $\mathcal{P}$-internal type $\operatorname{tp}(a/b)$ preserves $\mathcal{P}$-internality if for any tuple c such that $\operatorname{tp}(b/c)$ is almost $\mathcal{P}$-internal, the type $\operatorname{tp}(a/c)$ is also almost $\mathcal{P}$-internal.

This notion was introduced as an analogy with complex geometry, namely Moishezon morphisms (see Section 2.4.2). In the theory CCM of compact complex manifolds, one usually considers internality to the complex projective line $\mathbb{P}$, as it is, up to non-orthogonality, the only non-locally modular strongly minimal set.

It is proved in [24], working in CCM, that if X and Y are irreducible compact complex spaces and $f : X \to Y$ is a Moishezon morphism, then if a is a realization of the generic type of X, the type $\operatorname{tp}(a/f(a))$ preserves $\mathbb{P}$-internality. However, in the same paper, Moosa points out that the converse is false. Hence preserving internality is strictly weaker than being Moishezon. We will return to compact complex manifolds later in this chapter.

Remark that not all almost $\mathcal{P}$-internal types preserve $\mathcal{P}$-internality. Here is an example from differentially closed fields:

Example 4.2.2. Consider a generic constant a, and $\alpha \in \delta\log^{-1}(a)$, generic in that definable set. Then $\mathrm{tp}(\alpha/a)$ is $\mathcal{C}$-internal, and $\mathrm{tp}(a/\emptyset)$ is $\mathcal{C}$ internal, but as we pointed out before, the type $\mathrm{tp}(\alpha/\emptyset)$ is well-know to be not almost $\mathcal{C}$-internal.

Therefore, preserving internality is strictly stronger than being internal. As it turns out, being uniformly relatively internal is even stronger:

Theorem 4.2.3. *Let $\mathrm{tp}(a/b)$ be an almost $\mathcal{P}$-internal type. If $\mathrm{tp}(a/b)$ is uniformly almost $\mathcal{P}$-internal (in the sense of definition 4.1.3), then it preserves $\mathcal{P}$-internality.*

Proof. Let c be a tuple such that $\mathrm{tp}(b/c)$ is almost $\mathcal{P}$-internal. By assumption, there is a tuple e such that $ab \in \mathrm{acl}(b, e, \mathcal{P})$, and we can assume that $e \mathop{\smile\!\!\!\!\!\!\mid\;} ab$. Moreover, we can pick a realization of $\mathrm{tp}(e/ab)$ independent from c over ab, which yields $e \mathop{\smile\!\!\!\!\!\!\mid\;} abc$.

Almost $\mathcal{P}$-internality of $\mathrm{tp}(b/c)$ yields a tuple $\bar{b}$ of realizations of $\mathrm{tp}(b/c)$, independent from b over c, such that $b \in \mathrm{acl}(\bar{b}, \mathcal{P})$. We can assume that $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid\;}_c abe$. Since we also have $e \mathop{\smile\!\!\!\!\!\!\mid\;} abc$, so $e \mathop{\smile\!\!\!\!\!\!\mid\;}_c ab$, forking calculus yields $e\bar{b} \mathop{\smile\!\!\!\!\!\!\mid\;}_c ab$. But $ab \in \mathrm{acl}(b, e, \mathcal{P})$ and $b \in \mathrm{acl}(\bar{b}, \mathcal{P})$, so $ab \in \mathrm{acl}(e, \bar{b}, \mathcal{P})$. Hence $\mathrm{tp}(ab/c)$ is almost $\mathcal{P}$-internal.

□

This implication is not strict. To find counterexamples, we will turn to differentially closed fields of characteristic zero. Let us first make the following observation, valid in any stable theory:

Observation 4.2.4. *Let $\mathcal{P}$ be a family of $\emptyset$-type definable sets, and let $(q, \pi), q \in S(\emptyset)$ be relatively $\mathcal{P}$-internal. Suppose that $\pi(q)$ is orthogonal to $\mathcal{P}$ and of Lascar rank one. Then for any $a \models q$, the type $\mathrm{tp}(a/\pi(a))$ preserves $\mathcal{P}$-internality.*

Proof. Let c be such that $\mathrm{tp}(\pi(a)/c)$ is almost $\mathcal{P}$-internal. There are two possibilities. If $\pi(a) \mathop{\smile\!\!\!\!\!\!\mid\;} c$, then $\mathrm{tp}(\pi(a)/c)$ is still orthogonal to $\mathcal{P}$, and thus the only way it can be almost $\mathcal{P}$-internal is to be $\mathcal{P}$-algebraic. Else, we get that $\mathrm{tp}(\pi(a)/c)$ is algebraic, thus again $\mathrm{tp}(\pi(a)/c)$ is $\mathcal{P}$-algebraic. Finally, if $\mathrm{tp}(\pi(a)/c)$ is $\mathcal{P}$-algebraic, we can conclude that $\mathrm{tp}(a/c)$ is almost $\mathcal{P}$-internal.

□

Luckily for us, the following result, due to Hrushovski and Itai [14], allows for the identification of many sets orthogonal to the set $\mathcal{C}$ of constants in DCF_0. We now work in a monster model $\mathbb{U}$ of DCF_0.

Theorem 4.2.5. *Let $f \in \mathcal{C}[X]$. If $\frac{1}{f}$ has a simple and a double pole, then the strongly minimal set defined by $x' = f(x)$ is orthogonal to $\mathcal{C}$.*

Now consider a definable set $X = \{x, x' = f(x)\}$, orthogonal to the constants, and its pullback $\delta^{-1}(X)$ under the derivative. Let q be the generic type of $\delta^{-1}(X)$. Then (q, δ) is relatively $\mathcal{C}$-internal, and by Observation 4.2.4, for any $a \models q$, the type $\mathrm{tp}(a/\delta(a))$ preserves $\mathcal{C}$-internality.

Question 4.2.6. *When is such a type* not *uniformly almost $\mathcal{C}$-internal?*

In the following, we will give two examples showing that both cases are possible, as well as state a necessary and sufficient criteria for uniform internality. Let us start with an example of non-uniform $\mathcal{C}$-internality:

Example 4.2.7. Consider the polynomial $f(x) = (x-1)^2 x(x+1)$. By Theorem 4.2.5, the definable set $Y = \{x' = f(x)\}$ is orthogonal to the constants. Moreover, if q is the generic type of $\delta^{-1}(Y)$, then for any $a \models q$, the type $\mathrm{tp}(a/\delta(a))$ is not uniformly almost $\mathcal{C}$-internal.

Proof. Assume, by way of contradiction, that it is uniformly almost $\mathcal{C}$-internal. Since the Galois group of a fixed fiber is a subgroup of $\mathbb{G}_a(\mathcal{C})$, the additive group of the constant field, a quick Galois-theoretic argument yields that it is actually uniformly $\mathcal{C}$-internal.

The type $\delta(q)$ is orthogonal to the constants, thus Lemma 4.1.12 applies. There is a tuple t such that $\mathrm{tp}(t/\emptyset)$ is $\mathcal{C}$-internal, and its binding group is the image, under a finite-to-one morphism, of the binding group $\mathrm{Aut}(\mathrm{tp}(a_1, \cdots, a_n/\delta(a_1), \cdots, \delta(a_n))/\mathcal{P})$. The latter group is a subgroup of $\mathbb{G}_a(\mathcal{C})^n$.

Moreover, there exists $a \models q$, independent from t over $\emptyset$, with $a \in \mathrm{dcl}(t, \delta(a), \mathcal{C})$. That is, the tuple a is in the differential field $\mathcal{C}\{\delta(a), t\}$ generated by $\delta(a)$ and t over $\mathcal{C}$.

Let K be the differential field $\mathcal{C}\{t\}$ generated by t over $\mathcal{C}$. Because $\delta(\delta(a)) = f(\delta(a))$, the differential field $K\{\delta(a)\}$ is generated, as a field, by $\delta(a)$. That is, we have $K\{\delta(a)\} = K(\delta(a))$.

Moreover, as $\delta(a)$ in orthogonal to the constants and K internal to the constants, we see that $\delta(a)$ is independent from K. Thus, the type of $\delta(a)$ over K is simply the type of an algebraically transcendental element x, satisfying $\delta(x) = f(x)$. The differential field $(K(\delta(a)), \delta)$ is therefore isomorphic to $(K(X), \delta)$ with $\delta(X) = f(X)$.

By choice of K, we know that $a \in K(\delta(a))$. Therefore, there must be a solution S, in $K(X)$, of the equation $S' = X$.

Recall that $K(X)$ embeds, as a differential field, into the Laurent series $K((X))$ over K, by extending the derivation in the natural way. Thus, we can write $S = \sum_{i=N}^{\infty} a_i X^i$, for some $N \in \mathbb{Z}$ and $a_i \in K$, with $a_N \neq 0$. Using the equation $S' = X$, we deduce the following, for each i :

$$
\begin{aligned}
a_i' + (i-3)a_{i-3} - (i-2)a_{i-2} - (i-1)a_{i-1} + ia_i &= 0 \text{ if } i \neq 1 \\
&= 1 \text{ if } i = 1
\end{aligned}
$$

Thus, if $N \neq 1$, we obtain $a_N' = -Na_N$. If $N \neq 0$, this equation implies that $\mathrm{tp}(a_N/\emptyset)$ is $\mathcal{C}$-internal, with binding group a subgroup of $G_m(\mathcal{C})$. Recall that a_N belongs to K, which is generated, over $\mathcal{C}$, by t. The binding group of $\mathrm{tp}(t/\emptyset)$ is an algebraic group, finite extension of a subgroup of $G_a(\mathcal{C})^n$, for some n. This is possible only if $\mathrm{Aut}(\mathrm{tp}(a_N/\emptyset)/\mathcal{C})$ is finite, i.e. $a_N \in \mathrm{acl}(\mathcal{C})$, which implies that $a_N \in \mathcal{C}$. In particular, we have $Na_N = a_N' = 0$, so $a_N = 0$, a contradiction.

Thus, we have determined that $N = 0$ or $N = 1$. In that case, we obtain that $a'_1 = 1 - a_1$. As in the previous paragraph, this implies that $\mathrm{tp}(a_1/\emptyset)$ is $\mathcal{C}$-internal with Galois group a subgroup of $G_m(\mathcal{C})$, and by the same technique, that $a_1 \in \mathcal{C}$. The only solution is $a_1 = 1$. Using this argument, we can prove, by induction on i, that $a_i \in \mathcal{C}$ for all $i > 0$.

Thus the $a_i, i \geq 1$ are constants, solution to the recurrence relation $(i-3)a_{i-3} - (i-2)a_{i-2} - (i-1)a_{i-1} + ia_i = 0$. One can show that the Laurent series $\sum_{i=0}^{\infty} a_i X^i$ is not a rational function in X. Thus, even though a solution to $S' = X$ exists in $K((X))$, there is no solution in $K(X)$. A fortiori, the tuple a does not belong to $K(\pi(a))$, contradicting our assumption of uniform $\mathcal{C}$-internality. □

The following example will give an example of uniform internality, and also an example of a type which is uniformly $\mathcal{C}$-internal but not $\mathcal{C}$-algebraic:

Example 4.2.8. Consider $X = \{x' = x^3 - x^2\}$, and q the generic type of $\delta^{-1}(X)$. Then for any $a \models q$, the type $\mathrm{tp}(a/\delta(a))$ preserves $\mathcal{C}$-internality, and is uniformly $\mathcal{C}$-internal. Moreover, it is not $\mathcal{C}$-algebraic.

Proof. We simply have to prove that there is some $e \in \mathbb{U}$ such that, for all $a \models q$, we have $a \in \mathrm{dcl}(\delta(a), e, \mathcal{C})$. Consider t satisfying $t' = 1$, and $b \in X$ generic. Then $\delta(\frac{-1}{b} + t) = b$, thus $\frac{-1}{b} + t \in \delta^{-1}(\{b\})$. As this fiber is a one-dimensional $\mathcal{C}$-vector space, this implies that $\delta^{-1}(\{b\}) \subset \mathrm{dcl}(b, t, \mathcal{C})$, what we needed.

To prove that this type is not $\mathcal{C}$-algebraic, we will use methods similar to the one of the previous example, and will leave most details to the reader.

Let $a \models q$, and $b = \delta(a)$. We have to show that $a \notin \mathrm{acl}(b, \mathcal{C})$. First note that since $\delta(a) = b$, this is equivalent, by Fact 2.4.21, to $a \in \mathrm{dcl}(b, \mathcal{C})$. Using orthogonality of $\delta(q)$ to $\mathcal{C}$, this can be shown to be equivalent to $\delta(S) = X$ having a solution in the differential field $\mathcal{C}(X)$, with $\delta(X) = X^3 - X^2$. Passing to the Laurent series $\mathcal{C}((X))$, one can show, by working with coefficients, that such a solution cannot exist.

□

To conclude our study of these examples, let us state the following criteria, contained in a forthcoming paper, joint with Rémi Jaoui and Anand Pillay:

Proposition 4.2.9 (Pillay, Jaoui, J.). *If $Y = \{x' = f(x)\}$ is orthogonal to the constants, for some polynomial $f \in \mathcal{C}[x]$, and q is the generic type of $\delta^{-1}(Y)$, then for $a \models q$, the type $\mathrm{tp}(a/\pi(a))$ is uniformly internal to the constants if and only if there exists $c \in \mathcal{C}$ such that $\frac{x-c}{f(x)}$ has no simple pole.*

The methods used in the proof are quite different than the ones employed here, and more elegant. However, they rely on the fibers being one-dimensional over $\mathcal{C}$. The method used in this dissertation, relying Galois groups, might generalize more easily to higher dimensional fibers.

In conclusion, we have the following chain of strict implications:

$$\begin{array}{c} \mathcal{P}\text{-algebraic} \\ \Downarrow \\ \text{Uniformly } \mathcal{P}\text{-internal} \\ \Downarrow \\ \text{Preserves } \mathcal{P}\text{-internality} \\ \Downarrow \\ \mathcal{P}\text{-internal} \end{array}$$

The rest of this chapter will be dedicated to the arrow from uniformly $\mathcal{P}$-internal to preserves $\mathcal{P}$-internality. More precisely, we will examine known instances of preservation of internality, and try to determine whether or not uniform relative almost internality is involved.

4.3 Tangent Bundles in DCF_0

Inspired by results of Chatzidakis, Moosa and Trainor [4], we first examine the case of differential tangent bundles. In that paper, the authors proved that if X is a finite dimensional differential algebraic variety in a differentially closed field, then internality to the constants is preserved by the generic type of the differential jet spaces to X at generic points. Thus we settle to determine whether this type is also uniformly almost $\mathcal{C}$-internal.

For a detailed construction of differential tangent bundles, we refer the reader to [32]. We will recall some facts that will be useful to us. For the rest of this section, we will work in a fixed monster model $\mathbb{U} \models \mathrm{DCF}_0$.

Let X be a finite dimensional differential-algebraic variety, we will denote its differential tangent space at $b \in X$ by $T_\partial(X)_b$, and the differential tangent bundle of X by $T_\partial(X)$. We invite the reader to consult Section 2.4.1 for definitions of differential algebraic varieties and their dimension (or [32] and [4]). Here are some facts that we will make use of:

Fact 4.3.1.

1. *If X is a finite dimensional differential-algebraic variety, then so is $T_\partial(X)$, and for any $b \in X$, the tangent space $T_\partial(X)_b$ is a finite dimensional $\mathcal{C}$-vector space. Moreover, if k is the field of definition of X, the dimension of this vector space is equal to* $\dim(a/k)$.
2. *Any tuple in $T_\delta(X)$ is a pair (v, b), where $b \in X$ and $v \in T_\delta(X)_b$. In particular, we have a projection $\pi : T_\partial(X) \to X$.*
3. *There is an $\emptyset$ definable section $s : X \to T_\partial(X)$ of π, such that $s(b) = (b, 0)$*
4. *$X \to T_\partial(X)$ is a product-preserving covariant functor on the category of finite dimensional differential-algebraic varieties.*
5. *If $f : X \to Y$ is a map of differential-algebraic varieties, then for any $a \in X$, the map $T_\delta(f) : T_\delta(X)_a \to T_\delta(Y)_{f(a)}$ is linear. If f is generically finite-to-one, it is an isomorphism.*

Let X be a finite dimensional differential-algebraic variety. From the equations defining X, one can derive the defining equations of $T_\partial(X)$. There are differential polynomials $P_1(x_1, \cdots, x_m), \cdots, P_n(x_1, \cdots, x_m)$ such that X is defined by the zero locus of $P_1, \cdots, P_n$. Let d be the highest order of the $P_1, \cdots P_n$. Then there are polynomials $Q_1, \cdots Q_n \in \mathbb{M}[x_{j,k}, 1 \leq j \leq m, 1 \leq k \leq d]$ such that for all $1 \leq i \leq n$, we have $P_i(x_1, \cdots, x_m) = Q_i(\partial^k(x_j), 1 \leq j \leq m, 1 \leq k \leq d)$.

Fact 4.3.2. *Let X be a differential-algebraic variety, defined over the constants. Using the previous notations, if $a = (a_1, \cdots a_m) \in X$, the tangent space $T_\partial(X)_a$ is the zero locus of the polynomials $S_i = \sum_{k,j} \frac{\partial Q_i}{\partial x_{j,k}}(a_1, \cdots, a_m)\partial^k(y_j)$, for $1 \leq i \leq n$.*

See [19] for a detailed explanation of that fact.

A consequence of these equations, pointed out to us by Omar Léon-Sanchez, is the following:

Proposition 4.3.3. *Let X be a differential-algebraic variety, defined over the constants. If $a \in X$, then $\partial(a) \in T_\partial(X)_a$.*

Proof. Apply ∂ to the equations defining X.

□

Hence, we obtain that:

Corollary 4.3.4. *If X is a one dimensional differential-algebraic variety defined over the constants, with $b \in X$ generic and $a \in T_\delta(X)_b$ generic, then $\text{tp}(a/b)$ is uniformly $\mathcal{C}$-algebraic, and in particular uniformly $\mathcal{C}$-internal.*

Proof. If $X \subset \mathcal{C}$, then also $T_\delta(X) \subset \mathcal{C}$, so the result is immediate.

If not, then $\delta(b) \neq 0$, hence $\delta(b)$ is a non-zero element of the one dimensional $\mathcal{C}$-vector space $T_\delta(X)_b$, thus a basis. Hence $T_\delta(X)_b \subset \text{dcl}(b, \mathcal{C})$, an in particular $a \in \text{dcl}(b, \mathcal{C})$, yielding uniform algebraicity.

□

In particular, this is true even is X is orthogonal to the constants. Another way to obtain uniform internality is to use a group structure on X. More precisely, Fact 4.3.1 (4) implies that:

Corollary 4.3.5. *If G is a differential-algebraic group, then so is $T_\partial(G)$, and the map $T_\partial(G) \to G$ is a group morphism.*

In that case, the differential tangent bundle will be uniformly internal:

Proposition 4.3.6. *Let G be a differential algebraic group, and $T_\partial(G)$ its differential tangent bundle. Let $b \in G$ be a generic point of G, and $a \in T_\partial(G)_b$ be generic in the tangent space of b. Then* $\mathrm{tp}(a/b)$ *is uniformly $\mathcal{C}$-internal.*

Proof. This follows from the fact that for any differential algebraic group G, its differential tangent bundle splits as $G \times T_\delta(G)_e$, where e is the identity of G.

Alternatively, one can prove this 'by hand', using that the group law of G lifts to a group law on $T_\delta(G)$, by functoriality of T_δ.

□

In a forthcoming paper with Anand Pillay and Rémi Jaoui, an example of a differential tangent bundle that is not uniformly internal to the constants will be given. Thus, even for this specific case, it is a stronger notion than preserving internality.

4.4 Moishezon Morphisms in CCM

For this section, we will work in a monster model $\mathbb{A}$ of CCM, the first order theory of compact complex varieties (meaning reduced irreducible complex analytic space).

An important sort in CCM is $\mathbb{P}$, the complex projective line. It can be shown that the induced structure on $\mathbb{P}$ is simply its algebraic variety structure, and that is is, up to non-orthogonality, the only rank one non-locally modular definable set in CCM. Thus, it is natural to consider internality to $\mathbb{P}$.

Let us consider a complete type $p = \mathrm{stp}(a/b)$, for some tuples a and b. As was explained in Section 2.4.2, if we let $X = \mathrm{loc}(a, b)$ and $Y = \mathrm{loc(b)}$, then p is the generic type of the generic fiber of the map $f : X \to Y$. In [24], Moosa gives the following characterizations:

- p is $\mathbb{P}$ algebraic if and only if X meromorphically embeds into $Y \times \mathbb{P}_n$ over Y, for some $n \geq 0$.
- p is almost $\mathbb{P}$-internal if and only if p is $\mathbb{P}$-internal if and only if there is a complex analytic space $Y' \to Y$, such that the fibred product $X \times_Y Y'$ meromorphically embeds into $Y' \times \mathbb{P}_n$ over Y', for some $n > 0$.

Preservation of internality was introduced in that paper, with the goal of transposing a property of holomorphic maps, called being Moishezon, to a more general ω-stable context. It lies in between algebraicity and internality. To define it, we will follow the exposition given in [22].

Let S be a compact complex variety and $\mathcal{O}$ it structure sheaf, i.e. $\mathcal{O}$ is locally isomorphic to the sheaf of holomorphic functions.

A coherent analytic sheaf $\mathcal{F}$ on S is a sheaf of $\mathcal{O}$-modules such that there is a covering $\mathcal{X}$ of S by *euclidian* open sets and for any $U \in \mathcal{X}$, we have a resolution:

$$\mathcal{O}_U^p \xrightarrow{\alpha} \mathcal{O}_U^q \longrightarrow \mathcal{F}_U \longrightarrow 0$$

where the sheaves are restricted to U. Given such a sheaf, we will sketch the construction of the projective linear space over S associated to $\mathcal{F}$. Let us first explain the intuition behind this construction. Note that $\mathcal{O}_U^q$ acts on $U \times \mathbb{P}_{q-1}$, because if $(f_1, \cdots, f_q) \in \mathcal{O}_U$ and $(x, y_1, \cdots, y_q) \in U \times \mathbb{P}_{q-1}$, we can define the action as $(x, f_1(x)y_1, \cdots, f_q(x)y_q)$. However, this action does no go down to an action of $\mathcal{F}_U$ on $U \times \mathbb{P}_{q-1}$, as once we quotient, it might no longer be well-defined. The projective linear space associated to $\mathcal{F}$ will be the largest space on which this action makes sense.

More precisely, since the morphism α is $\mathcal{O}_U$-linear, it can be represented by a $q \times p$ matrix $M = (m_{i,j})$, with $m_{i,j} \in \mathcal{O}_U$ for all i, j.

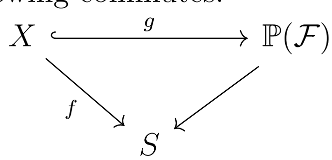
We let X be coordinates for U, and $(Y_1 : \cdots : Y_q)$ coordinates for $\mathbb{P}_{q-1}$. We (locally) construct $\mathbb{P}(\mathcal{F})_U$ as the analytic subset of $U \times \mathbb{P}_{q-1}$ defined by the equations

$$m_{1,i}Y_1 + \cdots + m_{q,i}Y_q = 0$$

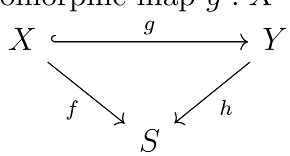
for all $i = 1, \cdots, p$. One can check that this does not depend on a the choice of coordinates for U, patches into $\mathbb{P}(\mathcal{F})$, and that the projections $U \times P_{q-1} \to U$ yield an holomorphic surjection $\mathbb{P}(\mathcal{F}) \to S$.

We can now give:

Definition 4.4.1. A morphism $f : X \to S$ of projective variety is *projective* if there is a coherent analytic sheaf $\mathcal{F}$ on S and an embedding $g : X \hookrightarrow \mathbb{P}(\mathcal{F})$ such that the following commutes:



Such a morphism is said to be *Moishezon* instead if it is bimeromorphic, over S, to a projective morphism. That is, there is a projective morphism $h : Y \to S$ and a meromorphic map $g : X \to Y$ such that the following commutes:



It is proved in [24] that

Proposition 4.4.2 (Moosa)**.** *If $f : X \to S$ is Moishezon, and $a \in X$ is generic, then* $\text{tp}(a/f(a))$ *preserves* $\mathbb{P}$*-internality, but the converse is false.*

On the other hand, if $f : X \to S$ is such that $\text{tp}(a/f(a))$ is $\mathbb{P}$-algebraic, then X meromorphically embeds into $S \times \mathbb{P}_n$, for some n, and thus f is Moishezon. However, there are Moishezon morphisms for which this is not the case.

Therefore, being Moishezon lies strictly between being $\mathbb{P}$-algebraic and preserving $\mathbb{P}$-internality. Recall that we proved that being uniformly internal also lies strictly between these two notions in general. The hope was therefore that, in CCM, being uniformly internal would exactly correspond to being Moishezon. It is unfortunately not the case.

In [22], Moosa proves:

Proposition 4.4.3. *Suppose X and S are compact complex varieties and $f : X \to S$ is a fibre space (i.e. its general fibres are irreducible). Then the following are equivalent:*

1. *For generic $a \in X(\mathbb{A})$, the type $\mathrm{tp}(a/f(a))$ is $\mathbb{P}$-internal*
2. *For some $n \geq 0$, there is a compact complex variety T and a surjection $T \to S$ such that $X_{(T)}$ is bimeromorphic to a subspace of $T \times \mathbb{P}_n$ over T*
3. *There is a compact complex variety T and a holomorphic surjection $T \to S$ such that $X_{(T)} \to T$ is Moishezon.*

where $X_{(T)}$ is an irreducible component of $T \times_S X$, projecting onto T.

We will prove:

Proposition 4.4.4. *Under the same assumptions, if $\mathrm{tp}(a/f(a))$ is uniformly $\mathbb{P}$-internal, then $\mathrm{tp}(a/f(a))$ is $\mathbb{P}$-algebraic. In particular f is Moishezon.*

Proof. Our proof is inspired of the proof of Proposition 4.4.3, $1 \Rightarrow 2$.

If $\mathrm{tp}(a/f(a))$ in $\mathbb{P}$-internal, then there is a tuple t containing $f(a)$ such that a is independent of t over $f(a)$, and a is interdefinable, over t, with a tuple from $\mathbb{P}(\mathbb{A})$ (see [22] for details).

If we let T be the locus of t, then the proof yields the following commutative diagram:

$$\begin{array}{ccccc} X & & X_{(T)} & \xrightarrow{h} & T \times \mathbb{P}_m \\ \downarrow{\scriptstyle f} & & \downarrow & \swarrow & \\ S & \longleftarrow & T & & \end{array}$$

where h is a meromorphism, and a bimeromorphism onto its image.

In the case where $\mathrm{tp}(a/f(a))$ is uniformly $\mathbb{P}$-internal, we can assume that $t = (f(a), \bar{b})$, where $\bar{b}$ is a tuple of realizations of $\mathrm{tp}(a)$, independent from a. If $Y \subset X^n$ is the locus of $\bar{b}$, then by independence, we have that $S \times Y$ is the locus of t.

So $T \times_S X = (S \times Y) \times_S X = Y \times X$, hence $X_{(T)} = Y \times X$ and we obtain the commutative diagram:

$$\begin{array}{ccccc} X & & Y \times X & \xrightarrow{h} & Y \times S \times \mathbb{P}_m \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle \mathrm{id}\times f}\big\downarrow & \swarrow & \\ S & \longleftarrow & Y \times S & & \end{array}$$

So for any $\overline{a} \in Y$, we have a diagram:

$$\begin{array}{ccc} \{\overline{a}\} \times X & \xrightarrow{h_{\overline{a}}} & \{\overline{a}\} \times S \times \mathbb{P}_m \\ {\scriptstyle \mathrm{id}\times f}\big\downarrow & \swarrow & \\ \{\overline{a}\} \times S & & \end{array}$$

There is $\overline{a}$ such that $h_{\overline{a}}$ is defined on a non-empty open subset of $\{\overline{a}\} \times X$. Indeed, the function h itself is defined on an open set of $Y \times X$, so on an open set of $\{\overline{a}\} \times X$, for each $\overline{a}$. If all these open sets are empty, then h is not defined anywhere, a contradiction.

Hence there is a meromorphic map:

$$\begin{array}{ccc} X & \xrightarrow{h_{\overline{a}}} & S \times \mathbb{P}_m \\ {\scriptstyle f}\big\downarrow & \swarrow & \\ S & & \end{array}$$

which is a bimeromorhism on its image because h is. Hence $f : X \to S$ is $\mathbb{P}$-algebraic!

□

This proposition is the expression, in CCM, of a much more general model-theoretic phenomenon, pointed out by Anand Pillay:

Proposition 4.4.5 (Pillay)**.** *Let M be a model of some stable theory T, eliminating imaginaries. Let $q \in S(M)$ and π be an M-definable function. Suppose that (q, π) is uniformly $\mathcal{P}$-internal, where $\mathcal{P}$ is a formula over M. Then (q, π) is $\mathcal{P}$-definable, i.e. for any $a \models q$, we have $a \in \mathrm{dcl}(M, \pi(a), \mathcal{P})$.*

This implies our result, as in CCM, any element of the standard model $\mathcal{A}$ is named by a constant, and thus $\mathrm{dcl}(\emptyset)$ is a model.

Note that, given a type $\mathrm{tp}(a/b)$, if it is stationary, the coordinate map $\pi : \mathrm{loc}(a) \times \mathrm{loc}(b) \to \mathrm{loc}(b)$ is a fibre space, by Proposition 2.4.27. Thus the previous proposition applies, and we have proved:

Proposition 4.4.6. *In CCM, any type* $\mathrm{tp}(a/b)$ *that is uniformly* $\mathbb{P}$*-internal is also* $\mathbb{P}$*-algebraic.*

As previously stated, there are Moishezon morphisms that are not $\mathbb{P}$-algebraic. Hence, being uniformly internal is a lot stronger than being Moishezon. The search for a model-theoretic property coinciding with Moishezoness in CCM is therefore still ongoing.

CHAPTER 5

AROUND THE CANONICAL BASE PROPERTY

This final chapter is devoted to the study of the canonical base property (CBP for short), and will compile different results I have obtained. This is a notion appropriate for the study of supersimple theories. In this dissertation, we will focus on superstable theories. Recall the definition of the canonical base property:

Definition 5.0.1. Let T be a superstable theory and $\mathbb{M} \models T$ a monster model. It is said to have the canonical base property if (possibly working over some parameters) for any tuples $a, b \in \mathbb{M}$, if $\mathrm{stp}(a)$ has finite Lascar rank and $b = \mathrm{Cb}(\mathrm{stp}(a/b))$, then $\mathrm{tp}(b/a)$ is almost $\mathcal{P}$-internal, where $\mathcal{P}$ is the family of non-locally modular rank one types.

Let us give some context to this definition. Intuitively, we expect the family of types $\mathcal{P}$ to be somewhat easy to understand, which is the reason why we study internality to this family. Indeed, in many concrete examples, this family reduces, up to non-orthogonality, to an algebraically closed field of characteristic zero (for example DCF_0 and CCM). This behavior is in line with the expectations set by Zilber's trichotomy.

Notice how the canonical base property can be thought of as one-basedness relative to $\mathcal{P}$. Indeed, a type p is one-based if for any realization $a \models p$ and any b, we have $\mathrm{Cb}(\mathrm{stp}(a/b)) \in \mathrm{acl}(a)$. The CBP replaces algebraicity by almost $\mathcal{P}$-internality.

Finally, let us give a visual interpretation of the CBP. If $b = \mathrm{Cb}(\mathrm{stp}(a/b))$, then $\mathrm{tp}(b/a)$ can be though of as parametrizing a family of "curves", all containing a

as a generic point. The canonical base property then states that this family of curves is parametrized (using some extra parameters) by a set definable in $\mathcal{P}$. In most examples, $\mathcal{P}$ reduces to an algebraically closed field, yielding that the family is parametrized by an algebraic variety.

The CBP has been used to prove various results in specific theories. See [32] and [2] for examples.

5.1 The Counterexample

For a while, it was believed that all superstable theories of finite Lascar rank had the CBP. However, Hrushovski, Palacín and Pillay constructed a theory without the CBP in [16], which still is the only known superstable counterexample.

Let us remind the reader how they construct such a structure. It is a reduct of an algebraically closed field of characteristic zero, constructed using tangent bundles. Namely, their structure consist of:

- A sort for the field of complex numbers $\mathbb{C}$, equipped with its field structure
- A sort $\mathcal{S} = \mathbb{C} \times \mathbb{C}$, equipped with:
  - The projection $\pi : \mathcal{S} \to \mathbb{C}$ on the first coordinate
  - The action of $\mathbb{C}$ on $\mathcal{S}$ defined by $c * (a, b) = (a, b + c)$
  - For each irreducible variety W of K^n, defined over $\mathbb{Q}^{\text{alg}}$, a predicate P_W for $TW \subset (\mathbb{C} \times \mathbb{C})^n$

They prove that this structure is $\aleph_1$-categorical and has Morley rank 2. Moreover, every type is analysable in $\mathbb{C}$, hence the family $\mathcal{P}$ of non locally-modular rank one types reduces, up to non-orthogonality, to the generic type of $\mathbb{C}$. By using Morley rank arguments, as well as studying the automorphism group $\text{Aut}(\mathcal{S}/\mathbb{C})$, they show:

Theorem 5.1.1 (Hrushovski, Palacín,Pillay). *The structure* $(\mathcal{S}, \mathbb{C})$ *in the prescribed language does not have the CBP.*

Remark that one could mimic that construction, but replacing $\mathbb{C}$ by an uncountable algebraically closed field of characteristic $p > 0$. In their paper, the authors ask if this structure has the canonical base property. We answer that question positively. To prove this, we will proceed through a lemma, valid regardless of the characteristic of the base field:

Lemma 5.1.2. *Let $(\mathcal{S}, F)$ be a structure consisting of an uncountable algebraically closed field, and another sort $\mathcal{S}$ constructed exactly as in the counterexample. Then the automorphism group $\mathrm{Aut}(\mathcal{S}/F)$ of automorphisms of this structure, fixing F pointwise, is isomorphic to $\mathrm{Der}(F)$, the group of derivations of F. In particular, if $\mathrm{Char}(F) > 0$, this group is trivial.*

Proof. In [16], it is proved that any $\delta \in \mathrm{Der}(F)$ gives rise to an element $\mathrm{Aut}(\mathcal{S}/F)$, by setting $\sigma_\delta(a, b) = (a, b+\delta(a))$. This comes down to a straightforward computation showing this preserves tangent bundles.

To go the other direction, we will prove that any element $\sigma \in \mathrm{Aut}(\mathcal{S}/F)$ arises this way. So fix $\sigma \in \mathrm{Aut}(\mathcal{S}/F)$.

First, notice that σ has to fix the first coordinate, as it fixes F and the projection on the first coordinate is in our language. For any $a, b \in F$, let $f(a, b)$ be such that $\sigma((a, b)) = (a, b + f(a, b))$, we need to prove that $f(a, b)$ does not depend on b, and is a derivation on the a coordinate.

Because σ respects the action of F, for any $a, b, c \in F$, we have:

$$\begin{aligned}
\sigma((a, b + c)) &= \sigma(c * (a, b)) \\
&= c * \sigma((a, b)) \\
&= c * (a, b + f(a, b)) \\
&= (a, b + c + f(a, b))
\end{aligned}$$

so in particular, the function $f(a,b)$ does not depend on b, we will denote $f(a,b) = f(a)$, for any b.

Let us now prove that $f(a+b) = f(a) + f(b)$ for any $a, b \in F$. Consider the variety $W \subset F^3$ given by the graph of addition, that is, by $x + y = w$. Its tangent bundle TW is also given by the graph of addition. That is, if $(a, b, a+b) \in W$, then $TW_{(a,b,a+b)}$ is given by $u+v = r$. Let $a, b \in F$, and $((a,u),(b,v),(a+b,u+v)) \in TW$, then $\sigma(((a,u),(b,v),(a+b,u+v))) \in TW$, yielding that:

$$((a, u + f(a)), (b, v + f(b)), (a + b, u + v + f(a + b))) \in TW$$

so $u + f(a) + v + f(b) = u + v + f(a + b)$, hence $f(a) + f(b) = f(a + b)$.

To show that for any $a, b \in F$, we have $f(ab) = af(b) + f(a)b$, we will proceed similarly, but this time using the graph of multiplication. So let $W \subset F^3$ be given by $w = xy$. If $a, b, ab \in W$, then the tangent bundle $TW_{(a,b,ab)}$ is given by $ub + va = r$. Let $a, b \in F$, and $((a,u),(b,v),(ab, ub + va)) \in TW$. Again using that σ preserves TW, we get that:

$$b(u + f(a)) + a(v + f(b)) = ub + va + f(ab)$$

which immediately yields $f(ab) = bf(a) + af(b)$.

This concludes the proof, as we have proved that $\sigma((a,b)) = (a, b + f(a))$ for any a, b, where f is a derivation on F.

Finally, if $\mathrm{Char}(F) > 0$, then because F is algebraically closed, there are no non-trivial derivations on F, giving us the last part of the Lemma.

□

Corollary 5.1.3. *If we consider the structure $(\mathcal{S}, F)$, with F an algebraically closed* *field of positive characteristic, then this structure does have the CBP.*

Proof. In fact, the previous Lemma yields much more: the sort $\mathcal{S}$ is in the definable closure of F. Since F has the CBP, this implies that the structure $(\mathcal{S}, F)$ has the CBP.

□

The authors also conjecture that any structure interpretable in an algebraically closed field of positive characteristic has the CBP. At the moment, this is still open, and we do not see how to attack this problem.

Let us give a bit more precision about how the structure $(\mathcal{S}, \mathbb{C})$ is proved to not have the CBP. The authors consider a tuple $(a, b, c, d) \in \mathbb{C}^4$ generic point of the algebraic variety W defined by $xw + yz = 1$, and $(u, v, r, s) \in \mathbb{C}^4$ such that $((a, u), (b, v), (c, r), (d, s))$ is a generic point of TW. As stated before, they then consider Morley rank and groups of automorphisms to prove:

- $((a, u), (b, v))$ is interalgebraic with $\mathrm{Cb}(\mathrm{stp}(((c, r), (d, s))/((a, u), (b, v))))$
- $\mathrm{tp}(((a, u), (b, v))/((c, r), (d, s)))$ is not almost $\mathbb{C}$-internal.

Because $\mathbb{C}$ is the only non-locally modular rank one type up to non-orthogonality, this proves that $(\mathcal{S}, F)$ does not have the canonical base property.

A careful examination of their proof leads one to believe that the field structure is not necessary. In fact, most of the argument relies on the machinery of general stability. For this reason, it seemed plausible that an "axiomatization" of the counterexample could be achieved. That is, a set of purely model-theoretic properties preventing a theory from having the canonical base property. This will be achieved in the next section.

5.2 Axiomatizing the Counterexample

Recall that the counterexample relies on generic points of the variety W given by $xw + yz = 1$, which are then lifted to the sort $\mathcal{S}$ using tangent bundles. The variety W is the seminal example of non-locally modular behavior in algebraically closed fields.

Examining the proof, one realizes that what is needed is not properties specific to tangent bundles, but the ability to transfer some independence and algebraicity between the sorts F and $\mathcal{S}$. This yields the following axiomatization:

Theorem 5.2.1. *Suppose we have a two sorted structure with sorts $\mathcal{P}$ and $\mathcal{S}$, with the sort $\mathcal{P}$ being strongly minimal, non-locally modular, and with geometric elimination of imaginaries. Moreover, assume there is a projection $\pi : \mathcal{S} \to \mathcal{P}$. Finally, suppose that:*

1. *The fibers of π are $\mathcal{P}$-internal, strongly minimal, and $\pi^{-1}(\{\pi(a)\}) \cap \mathrm{acl}(\pi(a)) = \emptyset$ for all $a \in \mathcal{P}$.*
2. *If $\pi(b_1), \cdots, \pi(b_m) \in \mathcal{P}$ are independent over the empty set, then $b_1, \cdots, b_m$ are independent over the empty set too*
3. *for any tuple $\bar{b}$ of elements of S, we have $\bar{b} \mathop{\smile\!\!\!\!\!\!\mid}_{\pi(\bar{b})} \mathcal{P}$*
4. *If $a_1, \cdots a_n \in \mathcal{P}$ and $a_n \in \mathrm{acl}(a_1, \cdots, a_{n-1})$, then there are $b_1, \cdots, b_n \in S$ such that $\pi(b_i) = a_i$ for all i and $b_n \in \mathrm{acl}(b_1, \cdots, b_{n-1})$*

then this structure is superstable of finite rank, $\aleph_1$-categorical, and does not have the canonical base property.

Proof. First note that this structure is $\mathcal{P}$-analysable because the projection π has $\mathcal{P}$-internal fibers, so in fact superstable of rank 2. Moreover, up to non-orthogonality, the set $\mathcal{P}$ is the only rank one set. This implies that this theory is unidimensional and $\aleph_1$-categorical, hence Morley rank will be additive by Lemma 2.1.29. Finally, internality to the family of non-locally modular rank one types is equivalent to internality to

$\mathcal{P}$. So we just need to find tuples a, b such that $\mathrm{Cb}(\mathrm{stp}(a/b)) = b$ and $\mathrm{tp}(b/a)$ is not almost $\mathcal{P}$-internal.

The proof will proceed using several claims. We encourage the reader to compare our proof to the proof in [16], as we are mostly abstracting what was done there.

General stability theory yields tuples $\overline{a} = (a_1, a_2) \in \mathcal{P}^2$ and $e \in \mathcal{P}^{\mathrm{eq}}$ such that $\mathrm{RM}(\overline{a}) = 2, \mathrm{RM}(\overline{a}/e) = 1$, and $e = \mathrm{Cb}(\mathrm{stp}(\overline{a}/e))$. Pick a Morley sequence $(\overline{a}_i)_{i\in\mathbb{N}}$ in $\mathrm{tp}(\overline{a}/e)$. Then we have:

Claim 5.2.2. *There is $n \in \mathbb{N}$ such that $\overline{a}_1, \cdots, \overline{a}_n$ are independent over the empty set and $e \in \mathrm{acl}(\overline{a}_1, \cdots, \overline{a}_n)$.*

Proof. It is always true that $e \in \mathrm{acl}((\overline{a}_i)_{i\in\mathbb{N}})$, and hence $e \in \mathrm{acl}(\overline{a}_1, \cdots, \overline{a}_n)$ for some n. If the $\overline{a}_i$ are independent over the empty set, we are done.

Else, let n be minimal such that $\mathrm{tp}(\overline{a}_{n+1}/\overline{a}_1, \cdots, \overline{a}_n)$ forks over the empty set. By assumption, we have $\mathrm{RM}(\overline{a}_{n+1}/\emptyset) = 2$, so $\mathrm{RM}(\overline{a}_{n+1}/\overline{a}_1, \cdots, \overline{a}_n) < 2$, but also $\mathrm{RM}(\overline{a}_{n+1}/\overline{a}_1, \cdots, \overline{a}_n, e) = 1$, so $\mathrm{RM}(\overline{a}_{n+1}/\overline{a}_1, \cdots, \overline{a}_n) \geq 1$. This allows us to conclude that $\mathrm{RM}(\overline{a}_{n+1}/\overline{a}_1, \cdots, \overline{a}_n) = 1$.

In particular, we obtained $\mathrm{RM}(\overline{a}_{n+1}/\overline{a}_1, \cdots, \overline{a}_n) = \mathrm{RM}(\overline{a}_{n+1}/\overline{a}_1, \cdots, \overline{a}_n, e)$, so $\overline{a}_{n+1} \mathop{\smile\!\!\!\!\!\!|}_{\overline{a}_1, \cdots, \overline{a}_n} e$. Thus $\mathrm{Cb}(\mathrm{stp}(\overline{a}_{n+1}/\overline{a}_1, \cdots, \overline{a}_n, e)) \in \mathrm{acl}(\overline{a}_1, \cdots, \overline{a}_n)$. Recall that $(\overline{a}_i)_{i\in\mathbb{N}}$ is a Morley sequence in $\mathrm{tp}(\overline{a}/e)$, so $\overline{a}_{n+1} \mathop{\smile\!\!\!\!\!\!|}_e \overline{a}_1, \cdots, \overline{a}_n$, yielding the equality $e = \mathrm{Cb}(\mathrm{stp}(\overline{a}_{n+1}/e)) = \mathrm{Cb}(\mathrm{stp}(\overline{a}_{n+1}/\overline{a}_1, \cdots, \overline{a}_n, e))$. Combining this with the previous algebraicity, we get $e \in \mathrm{acl}(\overline{a}_1, \cdots, \overline{a}_n)$, and by choice of n, the $\overline{a}_1, \cdots, \overline{a}_n$ are independent over the empty set, giving us the desired tuple.

□

Let us, for the rest of the proof, fix that n.

Geometric elimination of imaginaries gives us a tuple $\overline{c} \in \mathcal{P}$ interalgebraic with e. Fix such a $\overline{c} = (c_1, \cdots, c_m)$.

Since $\operatorname{tp}(\overline{a}_1) = \operatorname{tp}(\overline{a})$, we know that $\overline{a}_1 = (a_{1,1}, a_{1,2})$. Moreover $\operatorname{RM}(\overline{a}/c) = 1$, so we obtain that either $a_{1,1} \in \operatorname{acl}(\overline{c})$ or $a_{1,2} \in \operatorname{acl}(a_{1,1}, \overline{c})$. Indeed, if $a_{1,1} \notin \operatorname{acl}(\overline{c})$, then we have $1 = \operatorname{RM}(\overline{a}/c) = \operatorname{RM}(a_{1,1}/\overline{c}) + \operatorname{RM}(a_{1,2}/\overline{c}, a_{1,1}) = 1 + \operatorname{RM}(a_{1,2}/\overline{c}, a_{1,1})$. We can assume, without loss of generality, that $a_{1,2} \in \operatorname{acl}(a_{1,1}, \overline{c})$.

Assumption 4 then yields $\overline{b}_1 = (b_{1,1}, b_{1,2}), \overline{d} = (d_1, \cdots d_m)$ such that $\pi(\overline{b}) = \overline{a}_1$, $\pi(\overline{d}) = \overline{c}$ and $b_{1,2} \in \operatorname{acl}(b_{1,1}, \overline{d})$. We will prove that $\overline{d} = \operatorname{Cb}(\operatorname{stp}(\overline{b}_1/\overline{d}))$.

First, let us compute some ranks.

Claim 5.2.3. *We have* $\operatorname{RM}(\overline{b}_1) = 4$ *and* $\operatorname{RM}(\overline{b}_1/\overline{d}) = 2$.

Proof. As $\operatorname{RM}(\overline{a}_1) = 2$, we have $a_{1,1} \perp a_{1,2}$, thus $b_{1,1} \perp b_{1,2}$. This independence yields $\operatorname{RM}(\overline{b}) = \operatorname{RM}(b_{1,1}) + \operatorname{RM}(b_{1,2})$. Moreover, we have $\operatorname{RM}(b_{1,1}) = \operatorname{RM}(b_{1,1}/a_{1,1}) + \operatorname{RM}(a_{1,1}) = \operatorname{RM}(b_{1,1}) + 1$, and by Assumption 1. we have that $b_{1,1} \notin \operatorname{acl}(a_{1,1})$, so $\operatorname{RM}(b_{1,1}) = 2 = \operatorname{RM}(b_{1,2})$. Thus we conclude that $\operatorname{RM}(\overline{b}) = 4$.

For the other equality, first note that $a_{1,1} \perp \overline{c}$, as otherwise $a_{1,1} \in \operatorname{acl}(\overline{c})$, and because $a_{1,2} \in \operatorname{acl}(a_{1,1}, \overline{c})$, this would imply $\overline{a}_1 \in \operatorname{acl}(\overline{c})$, contradicting $\operatorname{RM}(\overline{a}_1/\overline{c}) = 1$. Hence, by Assumption 2, we obtain $b_{1,1} \perp \overline{d}$.

Now we can compute:

$$
\begin{aligned}
\operatorname{RM}(\overline{b}_1, \overline{d}) &= \operatorname{RM}(b_{1,1}, b_{1,2}, \overline{d}) \\
&= \operatorname{RM}(b_{1,1}, \overline{d}) \\
&= \operatorname{RM}(\overline{d}) + \operatorname{RM}(b_{1,1}/\overline{d}) \\
&= \operatorname{RM}(\overline{d}) + 2 \text{ since } b_{1,1} \perp \overline{d}
\end{aligned}
$$

On the other hand $\operatorname{RM}(\overline{b}_1, \overline{d}) = \operatorname{RM}(\overline{b}_1/\overline{d}) + \operatorname{RM}(\overline{d})$, thus $\operatorname{RM}(\overline{b}_1/d) = 2$.

□

This will help us obtain the following:

Claim 5.2.4. *Let $\overline{b}_1, \cdots, \overline{b}_n$ be a Morley sequence in* $\mathrm{tp}(\overline{b}_1/\overline{d})$*. The tuple $\overline{d}$ belongs to* $\mathrm{acl}(\overline{b}_1, \cdots \overline{b}_n)$*, where n is the number obtained in Claim 5.2.2.*

Proof. Let $\pi(\overline{b}_i) = a_i$. Since $\overline{a}_1, \cdots, \overline{a}_n$ is a Morley sequence in $\mathrm{tp}(\overline{a}_1/\overline{c})$, our choice of n yields that the $\overline{a}_i$ are independent over the empty set. By Assumption 2, the $\overline{b}_i$ are independent over the empty set too. We can then compute :

$$\begin{aligned} \mathrm{RM}(\overline{b}_1, \cdots, \overline{b}_n, \overline{d}) &= \mathrm{RM}(\overline{d}/\overline{b}_1, \cdots, \overline{b}_n) + \mathrm{RM}(\overline{b}_1, \cdots, \overline{b}_n) \\ &= \mathrm{RM}(\overline{d}/\overline{b}_1, \cdots, \overline{b}_n) + 4n \end{aligned}$$

and also :

$$\begin{aligned} \mathrm{RM}(\overline{b}_1, \cdots, \overline{b}_n, \overline{d}) &= \mathrm{RM}(\overline{d}) + \mathrm{RM}(\overline{b}_1, \cdots, \overline{b}_n/\overline{d}) \\ &= \mathrm{RM}(\overline{d}) + 2n \end{aligned}$$

which gives us $\mathrm{RM}(\overline{d}) - 2n = \mathrm{RM}(\overline{d}/\overline{b}_1, \cdots, \overline{b}_n)$. Recall that by Claim 5.2.2 we have $\overline{c} \in \mathrm{acl}(\overline{a}_1, \cdots, \overline{a}_n)$, thus:

$$\begin{aligned} 2n &= \mathrm{RM}(\overline{a}_1, \cdots, \overline{a}_n) \\ &= \mathrm{RM}(\overline{a}_1, \cdots, \overline{a}_n, \overline{c}) \\ &= \mathrm{RM}(\overline{a}_1, \cdots, \overline{a}_n/\overline{c}) + \mathrm{RM}(\overline{c}) \\ &= \mathrm{RM}(\overline{c}) + n \end{aligned}$$

yielding $\mathrm{RM}(\overline{c}) = n$. Strong minimality of the fibers implies that $\mathrm{RM}(\overline{d}) \leq 2n$, and since $\mathrm{RM}(\bar{d}) - 2n = \mathrm{RM}(\bar{d}/\bar{b}_1, \cdots, \bar{b}_n)$, this forces $\mathrm{RM}(\bar{d}/\bar{b}_1, \cdots, \bar{b}_n) = 0$.

□

Hence $\bar{d} \in \mathrm{acl}(\bar{b}_1, \cdots, \bar{b}_n)$, and $\bar{b}_1, \cdots, \bar{b}_n$ is a Morley sequence in $\mathrm{tp}(\bar{b}_1/\bar{d})$. This yields that $\bar{d}$ is interalgebraic with $\mathrm{Cb}(\mathrm{stp}(\bar{b}_1/\bar{d}))$. We will finish the proof by showing that $\mathrm{tp}(\bar{d}/\bar{b}_1)$ is not almost $\mathcal{P}$-internal. We will use the following general fact:

Claim 5.2.5. *Let $a \in S$ and $B \subset S$ be such that $\pi(a) \mathop{\smile\!\!\!\!\!\mid} \pi(B)$. Then $a \notin \mathrm{acl}(B \cup P)$.*

Proof. Let us assume, by way of contradiction, that $a \in \mathrm{acl}(B \cup \mathcal{P})$. Assumption 3 gives us $aB \mathop{\smile\!\!\!\!\!\mid}_{\pi(a)\pi(B)} \mathcal{P}$, which then yields $a \in \mathrm{acl}(\pi(a)B)$. But $\pi(a) \mathop{\smile\!\!\!\!\!\mid} \pi(B)$, which by Assumption 2 implies $a \mathop{\smile\!\!\!\!\!\mid} B$, hence also $a \mathop{\smile\!\!\!\!\!\mid}_{\pi(a)} B$. Finally, this implies $a \in \mathrm{acl}(\pi(a))$, contradicting Assumption 1.

□

We are now ready for the *coup de grâce*. Note that so far, we haven't used the non-local modularity of $\mathcal{P}$. It allows us to pick $\mathrm{tp}(\bar{a}_1/e)$ describing a *rich family of curves*, meaning $\mathrm{RM}(e) > 1$, or equivalently $e \notin \mathrm{acl}(\bar{a}_1)$. Because $\bar{c}$ is interalgebraic with e, and thanks to this choice for $\mathrm{tp}(\bar{a}/e)$, we have $\bar{c} \notin \mathrm{acl}(\bar{a}_1)$. Without loss of generality, we can assume that $c_1 \notin \mathrm{acl}(\bar{a}_1)$.

Assume that the structure does have the CBP. Then in particular, the type $\mathrm{tp}(\bar{d}/\bar{b}_1)$ would be almost $\mathcal{P}$-internal. As $c_1 \notin \mathrm{acl}(\bar{a}_1)$, we get $c_1 \mathop{\smile\!\!\!\!\!\mid} \bar{a}_1$, hence by Assumption 2 also $d_1 \mathop{\smile\!\!\!\!\!\mid} \bar{b}_1$. As $\mathrm{tp}(\bar{d}/\bar{b}_1)$ is almost $\mathcal{P}$-internal, so is $\mathrm{tp}(d_1/\bar{b}_1)$, implying the existence of a small set E such that $d_1 \mathop{\smile\!\!\!\!\!\mid}_{\bar{b}_1} E$ and $d_1 \in \mathrm{acl}(\bar{b}_1 \cup E \cup P)$. But $d_1 \mathop{\smile\!\!\!\!\!\mid} \bar{b}_1$, thus $d_1 \mathop{\smile\!\!\!\!\!\mid} \bar{b}_1 E$, so Claim 5.2.5 yields that $d_1 \notin \mathrm{acl}(\bar{b}_1 \cup E \cup P)$, a contradiction. Therefore, this structure does not have the CBP.

□

One might object that these conditions look very restrictive: what if such a structure simply does not exist? However, the counterexample exhibited by Hrushovski, Palacín and Pillay does have these properties, so at least one such structure exist. Hopefully, this characterization can pave the way for the construction of new counterexamples.

Note in particular that the only assumption made on the strongly minimal set $\mathcal{P}$ is non-local modularity, potentially allowing $\mathcal{P}$ to be CM-trivial, for example. Thus, a potential use of this theorem is to prescribe various properties of $\mathcal{P}$.

5.3 Final Remarks

In [2], Chatzidakis proves that if $b = \mathrm{Cb}(\mathrm{stp}(a/b))$ and $\mathrm{stp}(a/b)$ has finite Lascar rank, then $\mathrm{tp}(b/a)$ is always $\mathcal{P}$-analysable. Hence the CBP reduces to the problem of the collapse of analysability into internality. This was the original motivation behind the search for the internality criteria of Theorem 3.3.22.

After having obtained this criteria, and studied uniform internality in the previous chapter, we are now back to our starting point. As of yet, the tools developed in Chapter 3 did not produce any new criteria regarding the CBP. In fact, it is likely that finer tools will be needed if these methods are to succeed, as the Delta groupoids of Chapter 3 are too coarse to detect the fine inter-fiber interaction at play in Theorems 5.1.1 and 5.2.1. Indeed, observe that in both cases, we needed to understand the fibers over non-independent points, which Delta groupoids fail to capture.

However, if one is ready to drop type-definability, and work with the full simplicial groupoid, it seems likely that Theorem 5.2.1 could be expressed in the language of groupoids. This could help construct new groupoid-based counterexamples.

Another interesting avenue for research are strengthenings of the CBP. Historically, the first one to be defined was:

Definition 5.3.1. Let T be a superstable theory and $\mathbb{M} \models T$ a monster model. It is said to have the uniform canonical base property (UCBP) if (possibly working over some parameters) for any tuples $a, b \in \mathbb{M}$, if $\mathrm{stp}(a/b)$ has finite Lascar rank and $b = \mathrm{Cb}(\mathrm{stp}(a/b))$, then $\mathrm{tp}(b/a)$ preserves $\mathcal{P}$-internality, where $\mathcal{P}$ is the family of non-locally modular rank one types.

A surprising result obtained by Chatzidakis in [4] is the following:

Theorem 5.3.2. *If T has the canonical base property, then it also has the uniform canonical base property.*

A natural question is:

Question 5.3.3. *It T has the canonical base property, and $b = \mathrm{Cb}(\mathrm{stp}(a/b))$ with $\mathrm{tp}(a/\emptyset)$ of finite Lascar rank, is $\mathrm{tp}(b/a)$ uniformly almost $\mathcal{P}$-internal ?*

Note that as stated, this question does not make sense. Indeed, in the definition of uniform internality, we required our family of type $\mathcal{P}$ to be over a *fixed* set of parameters. To remedy this, we will need to work in a special class of superstable theories, namely theories that are nonmultidimensional with respect to non-locally modular Lascar rank one types. This means that any non-locally modular Lascar rank one type is non orthogonal to some $q \in S(\emptyset)$. Let us call this assumption $(*)$. As stated in [27], an example of theory not satisfying $(*)$ is given by an infinite family of algebraically closed sets, indexed by a set with extra structure. However, no natural counterexample are known.

Consider the family of types $\mathcal{Q}$ composed of types over $\emptyset$ that are almost $\mathcal{P}$-internal. It is proved in [27] that:

Proposition 5.3.4 (Palacín, Pillay)**.** *In a superstable theory satisfying $(*)$, a stationary type p is almost $\mathcal{Q}$-internal if and only if it is almost $\mathcal{P}$-internal.*

Thus under assumption $(*)$, one can study the CBP by considering the family $\mathcal{Q}$, which is over $\emptyset$. The pertinent question, in this context, is:

Question 5.3.5. *Let T be a superstable theory, satisfying the canonical base property and assumption $(*)$. Let $b = \mathrm{Cb}(\mathrm{stp}(a/b))$ with $\mathrm{tp}(a/\emptyset)$ of finite Lascar rank, is $\mathrm{tp}(b/a)$ uniformly almost $\mathcal{Q}$-internal ?*

We can thus define a new strengthening of the CBP as follows:

Definition 5.3.6. Let T be a superstable theory satisfying $(*)$ and $\mathbb{M} \models T$ a monster model. It is said to have the collapsed canonical base property (CCBP) if (possibly working over some parameters) for any tuples $a, b \in \mathbb{M}$, if $\mathrm{stp}(a/\emptyset)$ has finite Lascar rank and $b = \mathrm{Cb}(\mathrm{stp}(a/b))$, then $\mathrm{tp}(b/a)$ is uniformly almost $\mathcal{Q}$-internal.

and the questions becomes whether the CCBP is equivalent to the CBP. This turns out to be connected to yet another strengthening, defined by Palacín and Pillay in [27].

Definition 5.3.7. A theory T satisfying $(*)$ is said to have the strong canonical base property (SCBP) if for any a, b such that $\mathrm{tp}(a/\emptyset)$ has finite rank and $b = \mathrm{Cb}(\mathrm{stp}(a/b))$, we have $b \in \mathrm{acl}(a, \mathcal{Q})$, i.e $\mathrm{tp}(b/a)$ is $\mathcal{Q}$-algebraic.

We have the following (thanks to Anand Pillay for pointing this out):

Proposition 5.3.8. *In a superstable theory satisfying* $(*)$*, a type* $\mathrm{tp}(a/b)$ *is uniformly almost* $\mathcal{Q}$*-internal if and only if it is* $\mathcal{Q}$*-algebraic.*

Proof. The right to left direction is immediate. For the other direction, recall that by Theorem 4.1.11, if $\mathrm{tp}(a/b)$ is uniformly almost-$\mathcal{Q}$-internal, it is $\mathcal{Q}_{\mathrm{int}}$-algebraic, where $\mathcal{Q}_{\mathrm{int}}$ is the family of types $q \in S(\emptyset)$ that are $\mathcal{Q}$-internal. One can easily show that $\mathcal{Q} = \mathcal{Q}_{\mathrm{int}}$, yielding the result.

□

In particular we have obtained:

Corollary 5.3.9. *A superstable theory satisfying* $(*)$ *has the SCBP if and only if it has the CCBP.*

This completes the picture. Indeed, it is known by results of Hrushovski, Palacín and Pillay ([16] and [27]) that the strong canonical base property is strictly stronger than the canonical base property.

To show this, they prove a theorem linking the SCBP and Galois groups. First recall:

Definition 5.3.10. An A-type-definable Galois group is said to be *rigid* if any type-definable connected subgroup is defined over $\mathrm{acl}(A)$

and their theorem:

Theorem 5.3.11 (Hrushovski, Palacín, Pillay)**.** *If a theory T satisfies $(*)$, then T has the strong CBP is and only if every binding group relative to $\mathcal{Q}$ is rigid.*

One can show in particular that a non-rigid such binding group exists in DCF_0, proving that it does not have the strong CBP. As it was known to have the CBP by earlier work of Pillay and Ziegler [32], this provides the desired example.

Let us summarize the situation: in this dissertation, we have considered three different strengthenings of internality, and to each of these, there is a corresponding strengthening of the canonical base property. We now have a complete picture of the situation:

$\mathcal{P}$-algebraic	Strong canonical base property
$\Downarrow$	$\Updownarrow$
Uniformly $\mathcal{P}$-internal	Collapsed canonical base property
$\Downarrow$	$\Downarrow$
Preserves $\mathcal{P}$-internality	Uniform canonical base property
$\Downarrow$	$\Updownarrow$
$\mathcal{P}$-internal	Canonical base property

where the properties on the top two right rows are defined under assumption $(*)$.

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